1.2.1 Single partial differential equations
Partial Differential Eqs.

Evans Preface:

a) PDEs not just in two variables \((x, y)\) or \((x, t)\) 
Often, can treat \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) without much additional effort.

b) Many interesting PDEs are nonlinear

c) Many PDEs do not have smooth solutions, or even continuous solutions. (just)
leads to: generalized or weak solutions.

d) PDE not a branch of functional analysis

e) Notational nightmare

f) Good theory almost as useful as exact solutions.
(1.1) \( F(\partial^k u(x), \partial^{k-1} u(x), \ldots, \partial u(x), u(x), x) = 0 \quad x \in U \subset \mathbb{R}^n \)

is called a \( k^{th} \) order PDE.

\( \partial^k u(x) = \) collection of \( k^{th} \) order partial derivatives of \( u \).

\[
\partial^\alpha u(x) = \left\{ \frac{\partial^\alpha u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right\},
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha_i \geq 0 \)

and \( \lvert \alpha \rvert = \alpha_1 + \alpha_2 + \cdots + \alpha_n \)

\[
\partial^\alpha u(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x)
\]

\( n = 2 \); \( \alpha = (2, 1) \);

\[
\partial^\alpha u = \frac{\partial^2 u}{\partial x_1^2} \cdot \frac{\partial u}{\partial x_2}
\]

\( \partial^3 u(x) = \left\{ \partial^3 u, \partial^2 u, \partial u \right\} \)

\( \partial^3 u(x) = \left\{ \partial^3 u, \partial^2 \partial x_1^2 u, \partial^2 \partial x_2^2 u, \partial^3 u \right\} \)
i) A $k^{th}$ order PDE is **linear** if of form,

$$\sum_{|x| \leq k} a_k(x) D^k u = f(x),$$

homogeneous if $f \equiv 0$,

non homogeneous if $f \not\equiv 0$.

ii) **Semilinear** if highest order derivatives appear linearly:

form 

$$\left( \sum_{|x| = k} a_k(x) D^k u \right) + a_0(D^{k-1} u, D^{k-2} u, ..., Du, u, x) = 0$$

iii) **Quasilinear** if

form 

$$\left( \sum_{|x| = k} a_k(D^{k-1} u, D^{k-2} u, ..., Du, u, x) D^k u \right)$$

$$+ a_0(D^{k-1} u, D^{k-2} u, ..., Du, u, x) = 0.$$ 

iv) **Fully nonlinear** if

of form (i.1) but not i), ii) or iii).
A given problem for a PDE (e.g. initial and/or boundary value problem) is well-posed if:

a) problem has a soln
b) soln is unique
c) soln depends continuously on given data

Raises various questions:

1) what is a soln?

If answer is a smooth function of \( x = (x_1, \ldots, x_n) \) that satisfies the PDE at every point, then we can't allow solutions of \( u_t + F(u)_x = 0 \) (interesting) which have shocks.
2.1 Transport equation
Transport Eqn is:

(1) \[ u_t + b \cdot Du = 0 \]

where \( b = (b_1, b_2, \ldots, b_n) \) is fixed (const.)

and \( Du = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right) \)

( also known as gradient or \( \nabla \) )

Suppose \( u(x, t) \) is a C' solution

(1) says, \( (b_1, b_2, \ldots, b_n, 1) \cdot \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t} \right) = 0 \)

Along lines in \( \mathbb{R}^{n+1} \), in direction \( (b_1, b_2, \ldots, b_n, 1) \), the function \( u \) should be constant.

Consider (for fixed \( x, t \)), the line \( (x + sb, t + s) \), \(-\infty < s < \infty\)

\[ = (x, t) + s(b, 1) \]
If given data for $u$, can use it:

$$
\begin{aligned}
\text{exp} \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + b \cdot Du = 0 \\
\quad \text{Initial condition:} \quad u = g \text{ on } \mathbb{R}^n \times \{t = 0\}
\end{array} \right.
\end{aligned}
$$

Since assuming $C'$ soln, $g$ must also be $C'$ on $\mathbb{R}^n$

To find $u(x, t)$, use $u(x, t) = u(x - tb, 0) = g(x - tb)$.

let $s = -t$, gives

$$(x, t) + s(b, 1) = (x - tb, t - t)$$

Can verify directly that $u(x, t) = g(x - tb)$ satisfies Transport eqn, and I.C.

If $g$ not $C'$? Ex: $g$ discontinuous, but $g$ is $C'$ on some hypersurfaces $\mathbb{R}^n$

Then: at points in $\mathbb{R}^n$, corresponding to "good" regions, the formula for $u$ still "works".
\[(l_1, l_2, \ldots, l_n, 1) \cdot \left( \frac{du}{\partial x_1}, \frac{du}{\partial x_2}, \ldots, \frac{du}{\partial x_n}, \frac{du}{dt} \right) = 0\]

**Diagram:***

- \(R^{n+1}\) line: \((x+sl, t+s\cdot l)\) \(-\infty < s < \infty\)
- \(s > 0\)
- \((x, t)\) \(x = (x_1, \ldots, x_n)\)
- \(s < 0\)

At \(s = -t\),
\[(x+sl, t+s) = (x-lt, 0)\]
Nonhomogeneous Transport Eqn

\[
\begin{aligned}
\begin{cases}
  \frac{\partial u}{\partial t} + b \cdot \nabla u &= f & \text{in } \mathbb{R}^n \times (0, \infty), \\
  u &= g & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{cases}
\end{aligned}
\]

Fix \( x, t \), let \( z(s) = u(x + sb, t + s) \).

Then:

\[
\frac{dz}{ds} = \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} (x + sb) \frac{\partial}{\partial s} (x_i + sb_i) \right) + \frac{\partial u}{\partial t} (t + s) \frac{\partial}{\partial s} (t + s)
\]

\[
= \left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} (x + sb) b_i \right) + \frac{\partial u}{\partial t} (t + s)
\]

by PDE

\[
= D u (x + sb) \cdot b + \frac{\partial u}{\partial t} (t + s) = f (x + sb, t + s)
\]

\[
z(0) - z(-t) = \int_{-t}^{0} \frac{dz}{ds} ds = \int_{-t}^{0} f(x + sb, t + s) ds
\]

let \( s' = t + s, \ ds' = ds \):

\[
z(0) - z(-t) = \int_{-t}^{0} f(x + s'b, t + s) ds'
\]

\[
z(0) - z(-t) = \int_{s' = 0}^{t} f(x + s'b, t + s) ds'
\]

\[
u(x, t) = \frac{u(x - tb, 0) + \int_{s' = 0}^{t} f(x + s'b, t + s) ds'}{g(2x - tb)}
\]

\[
\text{Note: special case of method of characteristics}
\]
2.2 Laplace’s equation
Laplace's eqn \[ (1) \quad \Delta u = 0 \]

Poisson's eqn \[ (2) \quad -\Delta u = f \]

where \[ \Delta u = \sum_{k=1}^{N} u_{\kappa_k \kappa_k} = \sum_{k=1}^{N} \frac{2u}{\partial \kappa_k^2} \]

would like: \( u: \overline{U} \to \mathbb{R} \), such that \( u(x) = \sum_{k} f_k g(x) \), \( x \in \partial U \)

Define: A function \( u \in C^2 \) such that \( u = 0 \) in \( U \), is \textbf{harmonic} on \( U \).
Try to solve \( \Delta u = 0 \), in case \( U = \mathbb{R}^n \).

Guess: \( u(x) = u(r) \), where \( r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} \).

\[
\frac{\partial^2 u}{\partial x_i^2} = \frac{1}{2} (x_1^2 + \cdots + x_n^2)^{-1/2} \cdot \frac{\partial}{\partial x_i} (x_1^2 + \cdots + x_n^2) = \frac{x_i}{r} \quad \text{for each} \quad i = 1, \ldots, n.
\]

For \( u(x) = u(r) \),

\[
\frac{\partial u}{\partial x_i} = \frac{1}{2} \cdot n \cdot \frac{x_i}{r} = \frac{x_i}{r}.
\]

Since \( u'(r) \) is a function of \( r \),

\[
\frac{\partial^2 u}{\partial x_i^2} = \frac{1}{2} \cdot n \cdot \frac{x_i}{r} \cdot \frac{x_i}{r} + u'(r) \frac{\partial^2 r}{\partial x_i^2}
\]

\[
= \frac{1}{n} - \frac{x_i^2}{r^2}
\]

\[
\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = u''(r) \sum_{i=1}^{n} \left( \frac{x_i^2}{r^2} \right) + u'(r) \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{x_i^2}{r^2} \right)
\]

\[
= \frac{u''(r)}{n} \cdot n^2 + u'(r) \left( \frac{n}{n} - \frac{1}{n} \cdot \frac{n^2}{n^2} \right) = u''(r) + u'(r) \left( \frac{n}{n} - \frac{1}{n} \right).
\]
\[ \Delta u = 0, \quad \text{so:} \quad v''(r) + \frac{n-1}{r} v'(r) = 0. \]

Assume \( v' \neq 0; \quad \frac{d}{dr} (\ln |v'|) = \frac{v''}{v'} = \frac{1-n}{r} \]

\[ \text{If } n = 2, \quad \frac{d}{dr} (\ln |v'|) = -\frac{1}{r} \quad \Rightarrow \quad \ln |v'| = -\ln r + C \]

\[ \ln (2|v'|) = C_0 \]

\[ 2v' = \pm C_1 \]

\[ v' = \frac{\pm C_1}{2} \]

\[ v = \pm C_1 \ln r + C_2 \]

\[ \text{If } n \geq 3, \quad \frac{d}{dr} (\ln |v'|) = \frac{1-n}{r} \quad \Rightarrow \]

\[ v = \frac{b}{r^{n-2}} + C \]

Check: \( v = b r^{2-n} + C \)

\[ v' = b (2-n) r^{1-n} + 0 \]

\[ \ln |v'| = \ln b + \frac{1-n}{n} \ln r \quad \text{and} \quad \frac{d}{dr} (\ln |v'|) = 0 + 0 + \frac{1-n}{r} \]
Define
\[ \Phi(x) = \begin{cases} \frac{-\frac{1}{2n} \ln |x|}{n} & n = 2 \\ \frac{1}{n(n-2) \kappa(n) |x|^{n-2}} & n \geq 3 \end{cases} \]

where: \( \kappa(n) = \text{volume of unit ball in } \mathbb{R}^n \)

Then (claim):
\[ u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy \]

is a solution of Poisson's eqn \(-\Delta u = f\).

Will show true for \( f \in C_c^2(\mathbb{R}^n) \), compact support.

So:
\[ -\Delta \Phi = \delta_0 \] in formal sense that
\[ -\Delta u(x) = \int_{\mathbb{R}^n} -\Delta \Phi(x-y) f(y) \, dy = \int_{\mathbb{R}^n} \delta_0(x-y) f(y) \, dy = f(x) \]

yet unjustified.
2.2.1 Fundamental solution
Aside 1. Sphere $B(0, r)$ in $\mathbb{R}^n$

\[ \text{center radius} \]

Has volume $V = \alpha(n) r^n$, where $\alpha(n) = \text{vol.}(B(0, 1))$.

And surface area $V' = n \alpha(n) r^{n-1}$.

Aside 2. Let $g \in C_c(\mathbb{R}^n)$, continuous compact support

Then: $g$ is uniformly continuous. For each $\epsilon > 0$, there is a $\delta = \delta(\epsilon)$ such that

$|g(z) - g(y)| < \epsilon$ for all $z, y \in \mathbb{R}^n$ with $|z - y| < \delta$.

Note: $\delta = \delta(\epsilon)$ doesn't depend on $z$ or $y$; only on $\epsilon$. 
Notation: \( \text{supp}(f) = \text{support of } f = \{ x \in \mathbb{R}^n \text{ such that } f(x) \neq 0 \} \).

If \( \text{supp}(f) \subseteq \{ |x| \leq 2\varepsilon \} \), then for \( |y| \geq 2\varepsilon \),

it follows \( f(x-y) = 0 \) for all \( x \) with \( |x| \leq 2\varepsilon \).

(since \( |x-y| \geq 2\varepsilon \))

Integrations by parts:

See Appendix C: thin \( x \to y \) \( u \leftrightarrow v \)

\( u, v \in C^1(\overline{U}); \quad \overline{U} \subset \mathbb{R}^n \):

\[
\int_{\overline{U}} u(y) \nabla v(y) \, dy = -\int_{\overline{U}} v(y) \nabla u(y) \, dy + \int_{\partial \overline{U}} u(y) v(y) \nu \cdot dS(y)
\]
Recall \[ u(x) = \int_{\mathbb{R}^n} \varphi(y) f(y-x) \, dy \] since \( f \in C_c^\infty(\mathbb{R}^n) \), we can replace \[ \int_{\mathbb{R}^n} \] with \( \int_{\text{finite region}} \), based on difference quotient arguments.

And \[ \Delta u(x) = \int_{\mathbb{R}^n} \varphi(y) \Delta_y f(y-x) \, dy \]

\[ \Delta = \Delta_x = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) \]

But \( f(y-x) = f(y_1-x_1, y_2-x_2, \ldots, y_n-x_n) \)

\[ \Rightarrow \frac{\partial}{\partial y_i} f(y-x) = -\frac{\partial}{\partial y_i} f(y-x) \]

and \[ \frac{\partial^2}{\partial y_i^2} f(y-x) = \frac{\partial^2}{\partial y_i^2} f(y-x) \]

\[ \Rightarrow \Delta_y f(y-x) = \Delta_y f(y-x), \quad \forall y = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} \]

Therefore \[ \Delta u(x) = \int_{\mathbb{R}^n} \varphi(y) \Delta_y f(y-x) \, dy \]
\[ 
\Phi(y) = \begin{cases} 
-\frac{1}{2\pi} \log |y| & n = 2 \\
\frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|y|^{n-2}} & n \geq 3 
\end{cases} 
\]

\[ 
|D\Phi(y)| \leq \frac{C}{|y|^{n-1}} 
\]

\[ 
|D^2\Phi(y)| \leq \frac{C}{|y|^{n}} 
\ ; \text{ recall } I_e = \int_{B(0,\epsilon)} \Phi(y) \Delta y \delta(y-x) \, dy 
\]

\[ 
|I_e| \leq \|D^2\Phi\|_{L^\infty} \cdot \int_{B(0,\epsilon)} |\Phi(y)| \, dy 
\leq \begin{cases} 
C \epsilon^2 |\log \epsilon| & n = 2 \\
C \epsilon^2 & n \geq 3 
\end{cases} 
\]

\[ 
I_e = \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial}{\partial y} (x-y) \, d\sigma(y) 
\]

\[ 
|I_e| \leq \|D\Phi\|_{L^\infty} \cdot \int_{\partial B(0,\epsilon)} |\Phi(y)| \, d\sigma(y) 
\leq \begin{cases} 
C \epsilon |\log \epsilon| & n = 2 \\
C \epsilon & n \geq 3 
\end{cases} 
\]

\[ 
|\text{the area of ball } B(0,\epsilon)| \leq C \epsilon^{n-1} 
\]
Recall \( \Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\pi^2 |x|^{n-2}} & n \geq 3 \end{cases} \)

Claim: \( u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy \) is a solution of 

\[ -\Delta u = f, \] provided \( f \in C_c^0(\mathbb{R}^n) \)

Then: Let \( u(x) \) be as above. Then

\[ \frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(x-y) \frac{\partial f}{\partial x_i}(x-y) \, dy \] is defined and continuous.

Consider \( \lim_{h \to 0} \frac{f(x+h \xi) - f(x)}{h} - \frac{\partial f}{\partial x_i}(x) = 0 \), each \( \xi \in \mathbb{R}^n \).

For each \( \xi = (1,0,\cdots,0) \) then (claim - Heine-Borel) there is a \( \delta = \delta(\xi) \), such that
\[
\left\vert \frac{f(z+he_i) - f(z)}{h} - \frac{\partial f}{\partial x_i}(z) \right\vert < \varepsilon \quad \text{for all } z, y = z + \varepsilon h \text{ such that } \varepsilon h = |h| < \delta h.
\]

\[
\left\vert \int_{\mathbb{R}^n} \varphi(y) \left[ \frac{f(x+he_i-y) - f(x-y)}{h} - \frac{\partial f}{\partial x_i}(x-y) \right] \, dy \right\vert 
\]

\[
\leq \int_{\mathbb{R}^n} \left\vert \varphi(y) \right\vert \, dy < \varepsilon
\]

\[
\lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \varphi(y) \left[ \frac{f(x+he_i-y) - f(x-y)}{h} \right] \, dy
\]

\[
= \int_{\mathbb{R}^n} \varphi(y) \frac{\partial f}{\partial x_i}(x-y) \, dy
\]

Also: since \( \frac{\partial f}{\partial x_i}(x-y) \) is continuous, and \( \frac{\partial f}{\partial x_i} \in C_c(\mathbb{R}^n) \),
\[
\frac{2u}{\partial x_i} \text{ is continuous.}
\]
Given $x$.

Choose $R$ such that:

\[ 1 < R < 2 \]

\[ R < R \]

\[ f(y - \kappa) = 0 \] if \[ |y| > 2R \]

Can replace $\int_{\mathbb{R}^n} f(y) \, dy$ with $\int_{B(0, 2R)} f(y - \kappa) \, dy$ in the above integrals.

Pick $\varepsilon$, $0 < \varepsilon < R$, let $U = B(0, 2R) - B(0, \varepsilon)$.

Then

\[ \Delta u(x) = \int_{B(0, 2R)} F(y) \Delta y f(y - \kappa) \, dy \]

\[ = \int_{B(0, \varepsilon)} F(y) \Delta y f(y - \kappa) \, dy + \int_{U} F(y) \Delta y f(y - \kappa) \, dy. \]
\[ \int u \Phi(y) \frac{\partial}{\partial y_i} \left( \frac{1}{y_i} f(x-y) \right) \, dy = - \int u \Phi(y) \frac{\partial}{\partial y_i} f(x-y) \, dy + \int u \Phi(y) \frac{\partial f(x-y) \cdot v_i \, dS(y)}{\partial y_i} \] for each \( i = 1, \ldots, n \)

Sum:

\[ \int u \Phi(y) \Delta y f(x-y) \, dy = - \int u \Phi(y) \nabla \cdot \nabla f \, dy + \int u \Phi(y) (\nabla f \cdot v) \, dS(y) \]

\( \nabla \cdot \frac{\partial}{\partial n} \) deriv. in outward normal
Then:
\[ \sum_{\Omega} \Phi(y) \frac{\partial^2}{\partial y_i^2} \Phi(x-y) \, dy = - \int_{\Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, dy \]
\[ + \int_{\partial \Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, y^i \, dS(y) \]

and:
\[ \sum_{\Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, dy = - \int_{\Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, dy \]
\[ + \int_{\partial \Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, y^i \, dS(y) \]

adding over \( i \):
\[ \sum_{\Omega} \Phi(y) \Delta y \Phi(x-y) \, dy = - \int_{\Omega} \sum_{\Omega} \Phi(y) \cdot D y \Phi(x-y) \, dy \]
\[ \underbrace{\sum_{\Omega} \Phi(y) \cdot D y \Phi(x-y) \, dy}_{K \varepsilon} \]
\[ + \int_{\partial \Omega} \Phi(y) \frac{\partial \Phi(x-y)}{\partial y_i} \, y^i \, dS(y). \]
\[ \Delta_x u(x) = \sum_{B(0, \epsilon)} \Phi(y) \Delta_y f(y-x) \, dy + \sum_{U} \Phi(y) \Delta_y f(y-x) \, dx \]

where \( U = B(0, 2R) - B(0, \epsilon) \)

\[ = I_\epsilon + K_\epsilon + L_\epsilon, \text{ where} \]

\[ K_\epsilon = -\sum_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \nu} f(x-y) \, dS(y), \quad L_\epsilon = \sum_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f(x-y)}{\partial \nu} \, dS(y). \]

\[ K_\epsilon = -\sum_{U} \Delta_y \Phi \cdot \nabla_y f(x-y) \, dy = -\sum_{U} \Delta_y \Phi \, f(x-y) \, dy + \sum_{\partial U} \Phi \frac{\partial f(x-y)}{\partial \nu} \, dS(y) \]

away from \( y = 0 \).

\[ \Delta_y \Phi = 0. \]
\[
K_e = - \int_{\partial B(0, \epsilon)} (\nabla \phi \cdot n) f(x-y) \, dS(y)
\]

\[
= - \int_{\partial B(0, \epsilon)} \left( -\frac{1}{\alpha(n)} \frac{y}{|y|^2} \cdot \frac{y}{|y|} \right) f(x-y) \, dS(y)
\]

\[
= - \frac{1}{\alpha(n)} \int_{\partial B(0, \epsilon)} f(x-y) \, dS(y)
\]

But \(\alpha(n) \epsilon^{n-1}\) is surface area of \(\partial B(0, \epsilon)\).

So:
\[
K_e = - \int_{\partial B(0, \epsilon)} f(x-y) \, dS(y), \quad \text{where } \int \text{ means } \text{"average value"}
\]

As \(\epsilon \to 0\), by continuity of \(f\),
\[
\lim_{\epsilon \to 0} \int_{\partial B(0, \epsilon)} f(x-y) \, dS(y) = f(x).
\]

In all:
\[
\Delta u(x) = \lim_{\epsilon \to 0} (I_\epsilon + K_e + L_\epsilon) = 0 \implies f(x) + 0.
\]

(for each \(\epsilon, \epsilon \to 0 \) small, \(\Delta u(x) = I_\epsilon + K_e + L_\epsilon, \text{ all }\)
2.2.2 Mean-value formulas
To establish mean value properties for \( u = u(x) \), harmonic in \( U \), \( U \subset \mathbb{R}^n \) open.

Recall
\[
\frac{1}{\text{area of } \partial B(x, r)} \int_{\partial B(x, r)} u(y) \, ds(y) = \int_{B(x, r)} u(y) \, dy
\]

also
\[
\frac{1}{\text{volume of } B(x, r)} \int_{B(x, r)} u(y) \, dy
\]

must have \( B(x, r) \subset U \), \( r \) small enough so this is true.

Claim 1: \( \int_{\partial B(x, r)} u(y) \, ds(y) \) and \( \int_{B(x, r)} u(y) \, dy = u(x) \).

\( \partial B(x, r) = u(x); \quad B(x, r) \)

\( U \)

\( B(x, r) = \{ y \mid |x - y| < r \} \)
Proof of part 1: \( u(x) = \int_{\partial B(x, r)} u(y) \, dS(y) \)

Let \( \phi(r) = \int_{\partial B(x, r)} u(y) \, dS(y) \).

Change variables: let \( z = \frac{y-x}{r} \) (holding \( x \) constant).

\( y \in \partial B(x, r) \iff (y-x) = r \iff |z| = 1. \)

\[
\phi(r) = \int_{\partial B(0, 1)} u(x + nz) \, dS(n)
\]

\[
\phi'(r) = \int_{\partial B(0, 1)} \left[ \frac{\partial}{\partial y_1} u \left( x + nz \right) + \cdots + \frac{\partial}{\partial y_n} u \left( x + nz \right) \right] \, dS(n)
\]

\[
= \int_{\partial B(0, 1)} Du(x + nz) \cdot z \, dS(n)
\]

\[
= \int_{\partial B(x, r)} Du(y) \cdot (\frac{y-x}{r}) \, dS(y)
\]

\( y = \frac{y-x}{|y-x|} \) is unit outward normal.
\[ 
\phi'(x) = \int_{\partial B(x, r)} \frac{\partial u}{\partial n} \, ds(y) 
\]

But by Green's formula, \( \int_{\partial B(x, r)} \frac{\partial u}{\partial n} \, ds = \int_{B(x, r)} \Delta u \, dx \)

\[
\forall \ y \in \partial B(x, r) \n\]

Appendix C: \( \int_{\partial B(x, r)} u \Delta u \, ds = \int_{B(x, r)} u \frac{\partial u}{\partial n} \, ds - \int_{B(x, r)} u \nabla u \cdot \nabla u \, dx \)

for \( u = 1 \),

\[
\int_{\partial B(x, r)} u \Delta u \, ds = \int_{B(x, r)} u \frac{\partial u}{\partial n} \, ds = 0. 
\]

\[ 
\phi'(x) = \frac{1}{\text{(surf area of } \partial B(x, r))} \int_{\partial B(x, r)} \frac{\partial u}{\partial n} \, ds(y) = \frac{1}{\text{(surf area of } \partial B(x, r))} \int_{B(x, r)} \Delta u \, dy 
\]

\[ 
= 0, \text{ since } u \text{ is harmonic.} 
\]
Since $\phi'(r) = 0$, $\phi(r) = \text{const.}$, independent of $r$.

$$\phi(r) = \lim_{t \to 0^+} \phi(t) = \lim_{t \to 0^+} \int_{dB(x,t)} u(y) \, dS(y) = u(x),$$

since $u$ continuous at $x$.

For each $r$ such that $B(x, r) \subset U$, we have

$$u(x) = \phi(r) = \int_{\partial B(x, r)} u(y) \, dS(y).$$

This shows 1.

2. $\int_{B(x, r)} u \, dy = \int_{s=0}^{r} \left( \int_{\partial B(x, s)} u \, dS \right) \, ds$

$$= \int_{s=0}^{r} \left( \text{area of } \partial B(x, s) \right) \int_{\partial B(x, s)} u \, dS \, ds$$

$$= \int_{s=0}^{r} \left( \alpha(n) s^{n-1} \right) u(x) \, ds = u(x) \int_{s=0}^{r} \frac{s^{n-1}}{\alpha(n)} \, ds = u(x) \frac{r^n}{n \alpha(n)} \int_{s=0}^{r} s^{n} \, ds$$

$$= u(x) \frac{r^n}{n \alpha(n)} \left[ \frac{s^{n+1}}{n+1} \right]_{s=0}^{s=r} = \frac{r^n u(x)}{n \alpha(n)} \left( \frac{r^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} \right) = \frac{n u(x)}{n+1} r^n.$$
\[ u(x) = \frac{1}{\alpha(n) \pi^{n/2}} \int_{B(x, n)} u \, dy = \int_{B(x, 1)} u \, dy. \]

This shows 2.
2.2.3 Properties of harmonic functions
Preliminaries. Maximum principle.

If $U$ open is connected if it cannot be decomposed into the disjoint union of two (or more) open sets.

$U$ pathwise connected if, for any pair $x_0, x_1 \in U$,
there is a function $\gamma(s)$, continuous in $S$,
such that $\gamma(0) = x_0$, $\gamma(1) = x_1$,
and for each $s$, $0 \leq s \leq 1$, $\gamma(s) \in U$.

Then (Strong Maximum Principle)
If $u \in C^2(U) \cap C(\overline{U})$ is harmonic in $U$, open, pathwise connected:

then: (i) $\max_{\overline{U}} u = \max_U u$

(ii) If there is $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u$,
then $u$ is constant on $U$. 
Properties of harmonic functions.

Theorem 4 (Strong Maximum Principle)

If \( u \in C^2(U) \cap C(\overline{U}) \) is harmonic in \( U \), where \( U \) open, pathwise-connected, \( \overline{U} \) bounded, then:

(i) \( \max_{\overline{U}} u = \max_{\partial U} u \)

and

(ii) If there is \( x_0 \in U \) such that \( u(x_0) = \max_{\overline{U}} u = M \), then \( u \) is constant \((= M)\) on \( U \).

Proof of (i), assuming (ii):

Either 1. there is an \( x_0 \in U \) such that \( u(x_0) = M \)

or 2. for all \( x \in U \), \( u(x) < M \).

Case 1. then by (ii) \( u(x) \equiv M \) on \( U \); but \( u \in C(\overline{U}) \Rightarrow u(x) \equiv M \) on \( \partial U \). (i) holds.

Case 2. since \( \overline{U} \) is closed \& bounded and \( u \in C(\overline{U}) \)

there is a point \( y \in \overline{U} \) such that \( u(y) = M \).

but \( u(x) < M \) for all \( x \in U \), so \( y \not\in U \); must be \( y \in \partial U \). (i) holds.
Proof of Theorem 4, (ii); assuming otherwise connected.

To show: if there is an \( x_0 \in U \) such that \( u(x_0) = M \), then \( u \) is constant (\( \equiv M \)) on \( U \).

Let \( \epsilon_0 > 0 \) such that \( B(x_0, \epsilon_0) \subseteq U \).

By then, \( u(x) = \frac{\int_{B(x, \epsilon_0)} u \, dy}{|B(x, \epsilon_0)|} \).

If \( \exists \) point within \( B(x_0, \epsilon_0) \) at which \( u(y_0) < M \), there is a smaller ball inside \( B(x_0, \epsilon_0) \) with \( u(y) < M \) inside the smaller ball.

It follows that \( \int_{B(x_0, \epsilon_0)} u \, dy < M \).

The equality, \( \int_{B(x, \epsilon_0)} u \, dy = M \) is true, only if \( u(y) \equiv M \), all \( y \in B(x_0, \epsilon_0) \).

\( u \equiv M \) on \( B(x_0, \epsilon_0) \).
Let \( z \in U \) and suppose for contradiction \( u(z) < M \).

there is a path \( x(s), \ 0 \leq s \leq 1 \)

such that \( x(0) = x_0, \ x(1) = z \),

and \( x(s) \) is continuous, \( 0 \leq s \leq 1 \),

and \( x(s) \in U \) for each \( s \);

Let \( f(s) = u(x(s)) \). Then \( f \) is continuous, \( 0 \leq s \leq 1 \).
\[ f(s) = u(x(s)) \]

Let \( \overline{s}_m = \sup \left\{ s \mid f(s) = M \text{ for } 0 \leq s \leq s_m \right\} \)

If \( 1 > \overline{s} > \overline{s}_m \) \( \overline{s} \) is not a lower bound for this set. \( \exists s < \overline{s} \) such that \( f(s) < M \).

Consider \( B(\overline{x}_m, \epsilon_m) \) where \( \overline{x}_m = x(\overline{s}_m) \);
\[ f(s) \equiv M \]
\[ u(x(s)) = M \]
\[ u(\overline{x}_m) = M \]
\[ \Rightarrow \Box \]

But if consider \( u(\overline{x}_m) = M \), it follows \( u \equiv M \text{ in } B(\overline{x}_m, \epsilon_m) \) contradiction.

\[ \therefore u(\overline{s}) = u(x_0) = M. \]
Implications:

then \( s: g \in C(\partial U), f \in C(U) \).

There exists at one solution \( u \in C^2(U) \cap C(\overline{U}) \)

of \( \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \).

\[ \rightarrow \]

Let \( u \) and \( \tilde{u} \) both be solutions:

\[ -\Delta \tilde{u} = f \text{ in } U \]

\( \tilde{u} = g \) on \( \partial U \).

Let \( w = u - \tilde{u} \). Then

\[ -\Delta w = 0 \text{ in } U, \quad w \text{ is harmonic on } U \]

\( w = 0 \) on \( \partial U \).

\[ \max_{\overline{U}} w \leq \max_{\overline{U}} \tilde{w} = 0, \quad \text{so } w \leq 0, \text{ all } x \in \overline{U}. \]

\( -\Delta \tilde{u} = 0 \text{ in } U, \tilde{u} \text{ is harmonic on } U \)

\( \tilde{u} = 0 \) on \( \partial U \).

\[ \max_{\overline{U}} \tilde{w} \leq \max_{\overline{U}} \tilde{w} = 0, \quad \tilde{w} \geq 0, \text{ all } x \in \overline{U}. \]

\( u - \tilde{u} \leq 0, \text{ all } x \in \overline{U}. \)

Let \( \tilde{w} = u - \tilde{u}. \) As above,

\[ -\Delta \tilde{w} = 0 \text{ in } U, \tilde{w} \text{ is harmonic on } U \]

\( \tilde{w} = 0 \) on \( \partial U \).

\[ \max_{\overline{U}} \tilde{w} \leq \max_{\overline{U}} \tilde{w} = 0, \quad \tilde{w} \leq 0, \text{ all } x \in \overline{U}. \]

\( \tilde{u} - u \leq 0, \text{ all } x \in \overline{U}. \)

Since \( u \leq \tilde{u} \) and \( \tilde{w} \leq u \) for each \( x \in \overline{U} \), \( u(x) = \tilde{u}(x), \text{ for } \overline{U} \).
Theorem 6. If $u \in C(U)$ satisfies
\[
u(x) = \int_{B(x, r)} u(y) \, dy \quad \text{for each} \quad B(x, r) \subset U,
\]
then $u \in C^\infty(U)$.

**Proof:**
\[
\rho_n(r) = \begin{cases} \frac{C}{r^{n+1}} & 0 \leq r < 1 \\ 0 & r \geq 1 \end{cases}
\]
\[
\lim_{n \to \infty} \frac{1}{r^{n+1}} = -\infty
\]
\[
\lim_{n \to \infty} r^{n+1} e^{-r^{n+1}} = 0
\]

Then $\rho_n(r)$ is continuous for $r \geq 0$.

Let $\rho(x) = \rho(1|x|)$, \quad $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, $x \in \mathbb{R}^n$.

Pick $C$ so that \[
\int_{\mathbb{R}^n} \rho(x) \, dx = 1;
\]
\[
\int_{B(0, 1)} \rho(x) \, dx = 1, \quad \text{and} \quad \rho(x) = 0 \quad \text{for} \quad |x| \geq 1.
\]

For any $\varepsilon > 0$, let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$; then \[
\int_{\mathbb{R}^n} \rho_\varepsilon(x) \, dx = \int_{B(0, 1)} \rho_\varepsilon(x) \, dx = 1.
\]
\[ U_\varepsilon = \{ y \in U | \text{dist}(y, \partial U) > \varepsilon \} \]

Here, \( \text{dist}(y, \partial U) = \inf \{ \| y - z \|, z \in \partial U \} \)

Claim:

Let \( u_\varepsilon(x) = \int_U \eta_\varepsilon(x-y) u(y) \, dy \).

Then: \( u_\varepsilon(x) \equiv u(x) \), for each \( x \in U_\varepsilon \).

\[ \eta_\varepsilon(x,y) > 0 \text{ if } \| y - x \| < \varepsilon \]

\[ \eta_\varepsilon(x,y) = 0 \text{ if } \| y - x \| \geq \varepsilon \]

\( B(x,\varepsilon) \) (i.e. \( y \in B(x,\varepsilon) \))
\[ u^\varepsilon(x) = \int \mathcal{H}_\varepsilon(x - y) u(y) \, dy \]

\[ = \int \mathcal{H}_\varepsilon(x - y) u(y) \, dy \]

\[ = \int_0^\varepsilon \left( \int_{\partial B(x, r)} \mathcal{H}_\varepsilon(x - y) u(y) \, dS(y) \right) \, dr \]

For each \( y \in \partial B(x, \varepsilon) \), \( \|x - y\| = \varepsilon \)

so \( \mathcal{H}_\varepsilon(x - y) = \mathcal{H}_\varepsilon(r) \), where \( \mathcal{H}_\varepsilon(r) = \frac{1}{\varepsilon} \mathcal{H}\left(\frac{r}{\varepsilon}\right) \),

\[ u^\varepsilon(x) = \int_0^\varepsilon \left( \mathcal{H}_\varepsilon(r) \int_{\partial B(x, r)} u(y) \, dS(y) \right) \, dr \]

\[ = \int_0^\varepsilon \left( \mathcal{H}_\varepsilon(r) \left( \text{area of } \partial B(x, r) \right) \int_{\partial B(x, r)} u(y) \, dS(y) \right) \, dr \]

\[ = \int_0^\varepsilon \mathcal{H}_\varepsilon(r) \left( \text{area of } \partial B(x, r) \right) u(x) \, dr \]

\[ = \int_0^\varepsilon \mathcal{H}_\varepsilon(r) \left( \text{area of } \partial B(x, r) \right) \, dr = u(x) \int_{B(0, \varepsilon)} \mathcal{H}_\varepsilon(y) \, dy \]
Claim: \( u^\varepsilon(x) \in C^1(U_{3\varepsilon}) \).

\[
\frac{u^\varepsilon(x+he_i) - u^\varepsilon(x)}{h} = \frac{1}{h} \int_\Omega \frac{N^\varepsilon(x+he_i-y) - N^\varepsilon(x-y)}{h} u(y) \, dy
\]

\( T \)

difference quotient

used to find \( \frac{\partial u^\varepsilon}{\partial x_i} \): \( \varepsilon_i = (1, 0, 0, \ldots, 0) \)

\( \Omega \) \( h \ll \varepsilon \), support \( \{ N^\varepsilon(x+he_i-y) \} = B^\varepsilon(x+he_i, \varepsilon) \)

\( \xi \) fixed, \( \eta \) free

\( \int_\partial B^\varepsilon(\varepsilon, 2\varepsilon) \)
2.2.3 Properties of harmonic functions
Theorem 7: Estimate on Derivatives.

Inductive step.

Assume for each $k = 1, \ldots, K-1$, inequalities

$$|D^\alpha u(x)| \leq \frac{C_k}{\lambda^{n+K}} \|u\|_{L^1(B(x, \eta))} \quad \text{all tree}$$

where

$$C_k = \left(\frac{2^{n+K} \eta^n}{\lambda} \right)^k, \quad k = 1 \leq 1$$

For $\beta$ such that $|\beta| = K-1$; i.e., $\beta_1 + \beta_2 + \ldots + \beta_n = K-1$

$$|D^\beta u(x)| \leq \frac{C_{K-1}}{\lambda^{n+K-1}} \|u\|_{L^1(B(x, 5\eta))}$$

Let $\alpha$ be such that $|\alpha| = \kappa$. Then $D^\alpha u = (D^\alpha u)_{\kappa, i}$,

where $i = 1, \ldots, \kappa$.

For $\alpha = (3, 1, 0, 1)$; write

$$\kappa \in \mathbb{R}^4, \quad \alpha = (2, 1, 0, 1) + (1, 0, 0, 0)$$

$$D^\alpha u = \frac{2^2}{3!} \frac{\partial^2 u}{\partial x_2^2} + \frac{2}{2!} \frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{2}{2!} \frac{\partial u}{\partial x_1} = D^\beta u_{\kappa, i}.$$
\[ |D^\alpha u(x)| = |D^\beta u_\alpha(x)| \leq \frac{\alpha^K}{K^n} \max_{x \in D\beta(x, \frac{2r}{K})} |D^\beta u(x)| \]

for \( x \in D\beta(x_0, \frac{2r}{K}) \), \( B(x, \frac{K-1}{K}r) \subset B(x_0, r) \subset U \).

By hypothesis, \( |D^\beta u_\alpha(x)| \leq \frac{C_{K-1}}{(C_r)^{n+K-1}} \|u\|_{L^\alpha(B(x_0, \frac{2r}{K}))} \)

\[ |D^\alpha u(x)| \leq \frac{\alpha^K}{K^n} \frac{C_{K-1}}{(C_r)^{n+K-1}} \|u\|_{L^\alpha(B(x_0, r))} \]

\[ \leq \frac{C_K}{r^{n+K}} \|u\|_{L^\alpha(U \setminus B(x_0, r))} \]

where \( C_K = \frac{(2^{n+1} r K)^K}{K^n} \) for \( K = |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \).
2.2.3 Properties of harmonic functions
By inductive assumption:

\[
\|D^k u\|_{L^\infty(\partial B(x, \frac{n}{K}))} = \sup_{x \in \partial B(x, \frac{n}{K})} |D^k u(x)|,
\]

where for each \(x \in \partial B(x, \frac{n}{K})\),

\[
2U \ D^k u(x) \leq \frac{C_{K-1}}{s} \|u\|_{L^1(B(x,s))},
\]

provided \(B(x,s) \subset U\).

Take \(s = \frac{n}{K} = \frac{n}{2^{k-1}} < \frac{n}{2}\). If \(y \in \partial B(x,s)\),

then \(|x_0 - y| \leq |x_0 - x| + |x - y| \leq \frac{n}{2} \).

\[
\text{Take } s = \frac{n}{K} = \frac{n}{2^{k-1}} < \frac{n}{2}.
\]

\[
\text{provided } B(x,s) \subset U.
\]
Since \( D^\beta u_{\kappa} \) is harmonic, i.e. \( \Delta D^\beta u_{\kappa} = \frac{\partial^2}{\partial \kappa^2} (\Delta u) = 0 \)

\[
D^\beta u_{\kappa} = \int_{B(\kappa, \sqrt{K})} D^\beta u_{\kappa}(x) \, dx.
\]

Consider

\[
\int_{\mathbb{R}^n} \tilde{u}_{\kappa} \tilde{v} \, dx = -\int_{\mathbb{R}^n} \tilde{u} \tilde{v}_{\kappa} \, dx + \int_{\partial \Omega} \tilde{u} \tilde{v} \, d\mathcal{H}^n
\]

Take \( \tilde{u}_{\kappa} = (D^\beta u)_{\kappa}, \quad \tilde{v} = 1 \)

\[
D^\beta u_{\kappa}(x_0) = \frac{1}{\text{vol} \, B(\kappa, \sqrt{K})} \left[ -\alpha + \int_{\partial B(\kappa, \sqrt{K})} D^\beta u \, v \, d\mathcal{H}^n \right], \quad |\alpha| \leq 1
\]

\[
\frac{1}{|D^\beta u_{\kappa}(x_0)|} \leq \frac{\text{area} \, \partial B(\kappa, \sqrt{K})}{\text{vol} \, B(\kappa, \sqrt{K})}. \quad \sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))} = \frac{\sqrt{n} K^{n/2}}{\sqrt{K}^{n/2}} \sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))}
\]

\[
\frac{n \kappa(\gamma)(\frac{\gamma}{K})^{n-1}}{\alpha(\kappa)(\frac{\kappa}{K})^{n}} \cdot \frac{\sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))}}{\sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))}} = \frac{\alpha(\kappa)}{\kappa(\gamma)(\sqrt{K})^{n}} \cdot \frac{\sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))}}{\sup_{\partial B(\kappa, \sqrt{K})} |D^\beta u|_{L^\infty(\partial B(\kappa, \sqrt{K}))}}
\]
2.2.3 Properties of harmonic functions
Lionville's Theorem:

Suppose \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) is harmonic and bounded.

Then: \( u \) is constant.

Proof: Let \( x_0 \in \mathbb{R}^n \), pick \( r > 0 \).

By then \( \nabla u \) with \( \alpha = (1, 0, \ldots, 0) \)

\[
| \frac{\partial u}{\partial \alpha} (x_0) | \leq C \frac{\| \nabla u \|_{L^2(B(x_0, r))}}{n^{1/2}} \quad \text{where} \quad C = \frac{2^{n+1}}{\alpha(n)}
\]

But \( \| \nabla u \|_{L^2(B(x_0, r))} \) = \( \int_{B(x_0, r)} |u| dx \) \leq \( \text{vol. } B(x_0, r) \) \( \max_{x \in B(x_0, r)} |u(x)| \)

\[
\leq \alpha(n) r^n \| u \|_{L^\infty(B(x_0, r))}, \quad \text{finite if} \quad u \quad \text{bounded}.
\]

\[
\therefore | \frac{\partial u}{\partial \alpha} (x_0) | \leq \frac{C}{r^{n-1}} \alpha(n) r^n \| u \|_{L^\infty(\mathbb{R}^n)}
\]

But \( r \) is arbitrary; taking \( r \) large, see \( | \frac{\partial u}{\partial \alpha} (x_0) | = 0 \).

Similarly, \( \frac{\partial u_i}{\partial x_i} (x_0) = 0 \) for \( i = 1, 2, \ldots, n \):

\( \therefore \nabla u = 0 \) at all \( x \)

\( \Rightarrow u \) is constant.
2.2.3 Properties of harmonic functions
Representation of solutions of \(-\Delta u = f\) in \(\mathbb{R}^n\) \(n \geq 3\).

Let \(f \in C_c^2(\mathbb{R}^n)\)

Then, any bounded solution of \(-\Delta u = f\) in \(\mathbb{R}^n\),

must have the form,

\[
    u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy + C, \text{ for some constant } C.
\]

**Proof:**

Recall \(\Phi(x) = \frac{1}{n(n-2) \alpha(n)} \cdot \frac{1}{|x|^{n-2}}\), \(n \geq 3\).

By then 1,

\[
    \tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy \text{ is a solution.}
\]

Since \(\tilde{u}\) is bounded for \(|x| > 1\) (say) and since \(f\) has compact support (and is therefore bounded) it follows \(\tilde{u}\) is bounded.

If \(u\) is another solution of \(-\Delta u = f\), then

\[
    w := u - \tilde{u} \text{ is harmonic in } \mathbb{R}^n \ (\Delta w = 0) \text{ and by Liouville's Theorem, } w = C = \text{constant: } u = \tilde{u} + C \text{ as claimed}.
\]
2.2.3 Properties of harmonic functions
Lemma (bound on $k^k$)

There is a constant $C$ such that for all $k = 1, 2, \ldots$

\[ k^k \leq C k! e^k. \]

Stirling's

\[ \lim_{k \to \infty} \frac{k^{k+1/2}}{k! e^k} = \frac{1}{\sqrt{2\pi}}. \]
\[ \text{Thus} \quad e = \frac{1}{\sqrt{2\pi}} \]

There is a $K > 0$ such that

\[ \left| \frac{k^{k+1/2}}{k! e^k} - \frac{1}{\sqrt{2\pi}} \right| < \frac{1}{\sqrt{2\pi}} \]

so \[ \frac{k^{k+1/2}}{k! e^k} < \frac{3/2}{\sqrt{2\pi}}, \quad k \geq K. \]

Let \[ C = \max \left\{ \frac{3/2}{\sqrt{2\pi}}, \max_{1 \leq k \leq K} \frac{k^{k+1/2}}{k! e^k} \right\}. \]

Then \[ \frac{k^{k+1/2}}{k! e^k} \leq C \quad \text{for all} \quad k, \]

\[ \Rightarrow k^k \leq C k! e^k \leq C k! e^k. \]
**Lemma (Multinomial Theorem)**

For \( x_1, \ldots, x_n \) real or complex,

\[
(x_1 + x_2 + \cdots + x_n)^k = \sum \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}
\]

where \( 1\alpha_1 = k = \alpha_1 + \alpha_2 + \cdots + \alpha_n \)

\( \alpha_i! = (\alpha_i!)(\alpha_i!) \cdots (\alpha_i!) \)

Generalizes \((x+y)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x^j y^{k-j}\)

(binomial theorem)

\[
= \sum \frac{k!}{\alpha_1! \alpha_2!} x_1^{\alpha_1} x_2^{\alpha_2}
\]

**Corollaries**:

For \( x_1 = x_2 = \cdots = x_n = 1 \),

\[(1+1+\cdots+1)^k = n^k = \sum \frac{k!}{\alpha_1!} \]

**Corollary**:

For each \( \alpha = (\alpha_1, \ldots, \alpha_n) \) such that \( 1\alpha_1 = k \), \( \frac{1\alpha_1!}{\alpha_1!} \leq n^k \)

so \( 1\alpha_1! \leq n^k \alpha_1! = n^{1\alpha_1} \alpha_1! \).
Lemmas:

**Multi-dimensional Taylor polynomial theorem**

Let \( f(s) = u(x_0 + s(x-x_0)) \).

Then:

\[
\begin{align*}
    f(s) &= \sum_{k=0}^{N-1} f^{(k)}(x_0) \frac{s^k}{k!} + f^{(N)}(t) \frac{s^N}{N!},
\end{align*}
\]

some \( t \) between \( 0 \) and \( 1 \).

For \( s = 1 \):

\[
\begin{align*}
    u(x) &= \sum_{k=0}^{N-1} \sum_{1 \leq l_1 = k}^{\infty} \frac{D^k u(x_0)(x-x_0)^k}{k!} + \sum_{1 \leq l_1 = N}^{\infty} \frac{D^k u(x_0 + t(x-x_0))(x-x_0)^k}{k!},
\end{align*}
\]

some \( t \) between \( 0 \) and \( 1 \).

\[
|R_N(x)| \leq C M \sum_{1 \leq l_1 = N}^{\infty} \left( \frac{2^{l_1} a_1^2}{n} \right)^N |x-x_0|^{l_1}
\]
2.2.3 Properties of harmonic functions
Thin 10 Analyticity:

If \( u \) is harmonic in \( U \), then \( u \) is analytic in \( U \).

Analytic means: at any \( \mathbf{x}_0 \in U \), there is a neighborhood (Ball) such that with this mold, \( u \) can be represented as it's convergent power series.

Apply Taylor's Thin with remainder to \( f(s) = u(\mathbf{x}_0 + s(\mathbf{x} - \mathbf{x}_0)) \):

Claim: \( \sum_{\alpha!} \frac{D^\alpha u(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha \) converges to \( u(\mathbf{x}) \)

\( \alpha! = \alpha_1! \alpha_2! \ldots \alpha_n! \), \( (\mathbf{x}_i - \mathbf{x}_0)_i, (\mathbf{x}_i - \mathbf{x}_0)_2, \ldots (\mathbf{x}_i - \mathbf{x}_0)_n \)

\( \alpha! = \alpha_1! \alpha_2! \ldots \alpha_n! \)

\( R_N(\mathbf{x}) = u(\mathbf{x}) - \sum_{k=0}^{N-1} \frac{1}{k!} (\mathbf{x} - \mathbf{x}_0)^k \sum_{\alpha!} \frac{D^\alpha u(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha \)

\( = \sum_{1 \leq |\mathbf{t}| = N} \frac{D^\alpha u(\mathbf{x}_0 + \mathbf{t}(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0)^\alpha}{\alpha!} \), some \( 0 \leq \mathbf{t} \leq 1 \).

For \( 1 \leq |\mathbf{t}| = N \), \( \alpha_1 + \alpha_2 + \ldots + \alpha_n = N \).
Recall

\[ u(x) = \sum_{k=0}^{\infty} \sum_{|x| = k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha \]

\[ + \sum_{|x| = N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!} \]

\[ R_N(x) \]

But:

\[ |D^\alpha u(y)| \leq \frac{C_N}{r^{n+N}} \|u\|_{L^1(B(y, r))} \]

At \( r = \frac{1}{4} \text{dist}(x_0, \partial U) \)

At each \( y \in B(x, r) \), \( B(y, r) \subseteq B(x, 2r) \subseteq U \)

\[ |D^\alpha u(y)| \leq \frac{C_N}{r^{n+N}} \|u\|_{L^1(B(x, 2r))} \]
Let \( M = \frac{1}{\alpha(n) n} \| u \|_{L^1(B(x_0, 2r))} \)

(recall \( \alpha_N = \left( \frac{2^{n+1} n N}{\alpha(n)} \right)^N \)).

Then

\[
|D^k u(y)| \leq M \left( \frac{2^{n+1} n}{\alpha} \right)^k \leq k! e^k, \text{ where } k \leq N
\]

\[
|R_N| \leq C M \left( \frac{2^{n+1} n}{\alpha} \right)^N (N! e)^N \sum_{|\alpha| = N} \frac{(x - x_0)^\alpha}{\alpha!}
\]

\[
\leq C M \left( \frac{2^{n+1} n e}{\alpha} \right)^N \sum_{|\alpha| = N} \frac{N! (x - x_0)^\alpha}{\alpha!} \leq \frac{N!}{\alpha!} \frac{N!}{\alpha!} \leq n^N
\]

Take \( |x - x_0| < \frac{2}{2^{n+2} n^3 e} \). There are at most \( n^N \) such terms.

\[
|R_N| \leq C M \left( \frac{2^{n+1} n e}{\alpha} \cdot \frac{x - x_0}{2^{n+2} n^3 e} \right)^N \leq C M \left( \frac{\alpha}{2^N} \right)^N = C M \frac{1}{2^N} \rightarrow 0 \text{ as } N \rightarrow \infty.
\]
Since for $x \in B(x_0, \frac{\epsilon}{2^{n+2} n^3 e})$,

$$\sum_{k=0}^{\infty} \sum_{|x| = k} \frac{D^k u(x_0)(x - x_0)^k}{x^k},$$

the Taylor series converges to $u(x)$,

$u(x)$ is analytic at $x_0$.

Since $x_0 \in U$ was arbitrary, $u(x)$ is analytic in $U$.

Thm 10.
2.2.4 Green’s function
To find representation of solution $u$, of
\[
\begin{cases}
  -\Delta u = f & \text{in } U \\
  u = g & \text{on } \partial U.
\end{cases}
\]

For $u \in C^2(U)$:
\[
\nabla \cdot \nabla y V_e = 0
\]

From Appendix C Green's formula
\[
\int_{V_e} u \Delta y - \nabla y \cdot \nabla u = \int_{\partial V_e} \left( u \frac{\partial y}{\partial n} - \nabla u \cdot \nu \right) ds(y)
\]

Fix $x$; $\Delta y = \frac{\partial^2 y}{\partial y_1^2} + \cdots + \frac{\partial^2 y}{\partial y_n^2}$; let $\Phi(y) = \Phi(y-x)$,

where
\[
\Phi(x) = \begin{cases}
  -\frac{1}{2\pi} \log |x| & a = 2 \\
  \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & a > 2
\end{cases}
\]

Let $n = \text{outward unit normal}$:
\[
\nabla V_e = \partial U \cup \partial B(x, \epsilon)
\]
For $y \in V_\epsilon$, $y \neq x$ so $\Phi(y-x)$ is defined, and $\delta_y \Phi(y-x) = 0$.

\[ \int_{V_\epsilon} -\Phi(y-x) \delta_y u(y) \, dy = \int_{\partial V_\epsilon} \left[ u(y) \frac{\partial \Phi(y-x)}{\partial y} - \Phi(y-x) \frac{\partial u}{\partial y} \right] \, ds(y) \]

Consider \( \int_{\partial B(x, \epsilon)} \Phi(y-x) \frac{\partial u}{\partial y} \, ds(y) \):

\[ \left| \int_{\partial B(x, \epsilon)} \Phi(y-x) \frac{\partial u}{\partial y} \, ds(y) \right| \leq \max_{y \in \partial B_\epsilon} \left\{ \Phi \right\} \cdot C \left( \frac{\text{area}}{\text{vol} \, \partial B_\epsilon} \right) \]

\( u \in C^2(U) \);

\[ \left| \frac{\partial u}{\partial y} \right| \leq \|D \, u\|_1 \leq C \].

\[ \therefore \left| \int_{\partial B_\epsilon} \Phi(y-x) \frac{\partial u}{\partial y} \, ds(y) \right| \leq C_2 \epsilon^{n-1} \left\{ \begin{array}{ll}
\frac{1}{2^k \log e} & \text{if } n = 2 \\
\frac{1}{n(n-2) \kappa(n)} \frac{1}{\epsilon^{n-2}} & \text{if } n \geq 3
\end{array} \right. \]

\( \rightarrow 0 \) as \( \epsilon \rightarrow 0^+ \).
In proof of Theorem 6.2.2,

\[ D\Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|} \nu, \quad y \neq 0; \]

and \( \nu = -\frac{y}{|y|} = -\frac{y}{\epsilon} \) on \( \partial B(0, \epsilon) \).

Here, replace \( \partial B(0, \epsilon) \) with \( \partial B(x, \epsilon) \), replace \( y \) with \( z = y-x \):

\[ D\Phi(y-x) = -\frac{1}{n\alpha(n)} \frac{z}{|z|} \nu, \quad z = y-x \neq 0 \quad \text{for} \quad y \in \partial B(x, \epsilon) \]

and \( \nu = -\frac{z}{|z|} = -\frac{y-x}{|y-x|} \) on \( \partial B(x, \epsilon) \).

\[ 2\Phi(y-x) = -\frac{4}{\epsilon} - \frac{z}{n\alpha(n)} \frac{z}{|z|} \nu = \frac{1}{\epsilon n\alpha(n)} \frac{1}{|z|^n} \]

\[ = \frac{\epsilon^{2-(n+1)}}{n\alpha(n)} = \frac{\epsilon^{1-n}}{n\alpha(n)}. \]

\[ \int_{\partial B(\epsilon, x)} u(y) \frac{D\Phi(y-x)}{\partial \nu} \, ds(y) = \int_{\partial B(\epsilon, x)} u(z) \frac{D\Phi(z)}{\partial \nu} \, ds(z) \]

\[ = \frac{1}{n\alpha(n)} \int_{\partial B(0, \epsilon)} u(z) \, ds(z) = \int_{\partial B(0, \epsilon)} u(z+g) \, ds(z), \quad g \in \partial B(0, \epsilon). \]
Since $u$ is continuous at $\kappa$, \[ \lim_{\varepsilon \to 0^+} \int_{B(\kappa, \varepsilon)} u(x + \varepsilon z) \, ds(z) = u(\kappa). \]

Recall
\[ \int_{V_\varepsilon} u(y) \frac{\partial \phi}{\partial y} (y - \kappa) - \Phi(y - \kappa) \Delta u(y) \, dy = \int_{\partial V_\varepsilon} \left[ u \frac{\partial \Phi}{\partial n}(y - \kappa) - \Phi(y - \kappa) \frac{\partial u}{\partial n}(y) \right] \, ds(y) \]

Taking $\lim_{\varepsilon \to 0}$ gives
\[ -\int_{V_\varepsilon} \Phi(y - \kappa) \Delta u(y) \, dy = \int_{\partial V_\varepsilon} \left[ u \frac{\partial \Phi}{\partial n}(y - \kappa) - \Phi(y - \kappa) \frac{\partial u}{\partial n}(y) \right] \, ds(y) \]

\[ + u(\kappa) = 0 \]

So
\[ u(\kappa) = \int_{\partial V_\varepsilon} \left[ \Phi(y - \kappa) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y - \kappa) \right] \, ds(y) \]

\[ - \int_{\partial V_\varepsilon} \Phi(y - \kappa) \Delta u(y) \, dy ; \quad \{ \text{only assumed on } \kappa \in C^2(\overline{\Omega}) \} \]
issue: don't know \( \frac{\partial u}{\partial y} (y) \) and \( u(y) \) for \( y = \partial u \).

Suppose we only have \( u(y) \) on \( \partial u \).

The term \( \int_{\partial u} \Phi(y-x) \frac{\partial u}{\partial y} (y) \, dS(y) \) poses a problem.

Introduce "corrector" function \( \phi^*(y) \):

\[
\left\{ \begin{array}{l}
\Delta_y \phi^*(y) = 0 \quad \text{in} \ U \\
\phi^*(y) = \Phi(y-x) \quad \text{for} \ y \in \partial U
\end{array} \right.
\]

Suppose this is possible, and \( \phi^*(y) \) is "well-behaved" at \( y = \partial u \).

Use \( \phi^*(y) \) instead of \( \Phi(y-x) \) in Green's formula:

\[
\int_{\partial u} \left[ u \left( \frac{\partial \phi^*}{\partial y} \right) (y) - \left( \frac{\partial u}{\partial y} \right) \phi^*(y) \right] \, dS(y)
\]

Add (25) and (27):

\[
\text{terms } \int_{\partial u} \Phi(y-x) \frac{\partial u}{\partial y} \, dS(y) \text{ and } -\int_{\partial u} \phi^*(y) \frac{\partial u}{\partial y} \, dS(y), \text{ cancel.}
\]

\[
(25) \quad u(x) = \int u(y) \left( \frac{\partial \Phi}{\partial y}(y-x) - \frac{\partial}{\partial y} \phi^*(y) \right) \, dS(y) - \int (\Phi(y-x) - \phi^*(y)) \Delta_y u \, dy
\]

\[
(27) \quad \frac{\partial u}{\partial y} (y) = \int u(y) \left( \frac{\partial \Phi}{\partial y}(y-x) - \frac{\partial}{\partial y} \phi^*(y) \right) \, dS(y) - \int (\Phi(y-x) - \phi^*(y)) \Delta_y u \, dy
\]
2.2.4 Green’s function
Recall last day

\[ u(x) = \int \Phi(y-x) \frac{\partial u(y)}{\partial n} - u(y) \frac{\partial \Phi(y-x)}{\partial n} \, d\Sigma(y) \]

and

\[ \int \frac{\partial}{\partial n} \left[ u \frac{\partial \Phi(y)}{\partial n} - \Phi(y) \frac{\partial u}{\partial n} \right] \, d\Sigma(y) \]

\[ = \int \frac{\partial}{\partial n} \left[ u \Phi(y) - \Phi(y) u(y) \frac{\partial \Phi(y)}{\partial n} \right] \, d\Sigma(y) \]

\[ - \int \Phi(y-x) \Delta_g u(y) \, dy \]

\[ = \int \frac{\partial}{\partial n} \left[ u \Phi(y) - \Phi(y) u(y) \frac{\partial \Phi(y)}{\partial n} \right] \, d\Sigma(y) \]

\[ = \int \frac{\partial}{\partial n} \left[ u \Phi(y) - \Phi(y) u(y) \frac{\partial \Phi(y)}{\partial n} \right] \, d\Sigma(y) \]

Let \( G(x, y) = \Phi(y-x) - \Phi^*(y) \).

Then

\[ u(x) = -\int u(y) \frac{\partial G(x, y)}{\partial n} \, d\Sigma(y) \]

\[ - \int G(x, y) \Delta_g u(y) \, dy \]

\[ \text{represents solution of Poisson PDE} \]

\[ \nabla u = 0 \]
Claim: \( G(x, y) = G(y, x) \).

Let \( V = U - B(x, \varepsilon) - B(y, \varepsilon) \)

Define \( \nu(z) := G(x, z) \),
\( \omega(z) := G(y, z) \)

will now show \( \nu(y) = \omega(x) \).

Recall \( \nu(g) = G(x, g) = \Phi(g-x) - \Phi'(g-x) 

so \( \nu(z) = 0 \), \( z \in \partial U \).

Similarly, \( \omega(z) = 0 \), \( z \in \partial U \).

Green's formula
\[
\int_V \nu \nabla w - w \nabla \nu \, dx = \int_{\partial V} \left( \nu \frac{\partial w}{\partial n} - w \frac{\partial \nu}{\partial n} \right) \, ds(z)
\]

\( \Delta_g \nu = 0 \) and \( \Delta_g w = 0 \)

in \( V \).

\[
\int_{\partial B(x, \varepsilon)} \left( \frac{\partial w}{\partial n} - \frac{\partial \nu}{\partial n} \right) \, ds(g) = \int_{\partial B(y, \varepsilon)} \left( \frac{\partial \nu}{\partial n} - \frac{\partial w}{\partial n} \right) \, ds(g)
\]
For $y$ near $x$, \( w(y) = G(y, z) = \frac{\Phi(z-y)}{x} f(y) \) is well-behaved
but \( v(z) = G(x, z) = \frac{\Phi(z-x)}{x} f'(x) \) blows up as $z \to x$.
because \( \Phi(z-x) \) blows up.

Here, \[
\int_{\partial B(x, \epsilon)} \frac{\partial w}{\partial \nu} dS(z) = \int_{\partial B(x, \epsilon)} (\frac{\partial}{\partial \nu} \Phi(z-x) - \Phi'(z)) w \, dS(z)
\]
well behaved
\[
= \int_{\partial B(x, \epsilon)} w(z) \, dS(z) + o(1)
\]
means, quantity that $\to 0$ as $\epsilon \to 0^+$

\[
\int_{\partial B(x, \epsilon)} \left( \frac{\partial w}{\partial \nu} \right) dS(z) = O(\epsilon) \to 0
\]
as $\epsilon \to 0^+$

Similarly, \[
\int_{\partial B(y, \epsilon)} \left( \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} \right) dS(z) = \int_{\partial B(y, \epsilon)} w(z) \, dS(z) + o(1)
\]

\[
\frac{\partial}{\partial \nu} \left( \Phi(z-y) - \Phi'(z) \right)
\]
So: \( w(x) = v(y) \), and so \( G(x, y) = G(y, x) \).

Basic symmetry of Green's functions.)

Not in general true that Green's functions are radial:

Be true that \( \varphi(y - x) \) is radial.

\[ \therefore \varphi(y - x) = \varphi(1y - x1) \]

meaning that \( \varphi(y - x) \) is a function of just \( 1y - x1 \).

\[ \forall x \in \mathbb{R}^2, \quad \varphi(y - x) = -\frac{i}{2\pi} \log |y - x| \]

\[ = -\frac{i}{2\pi} \log \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \]

\[ \forall n \geq 3, \quad \varphi(y - x) = \frac{1}{n(n - 2)\pi(n)} \frac{1}{|y - x|^{n - 2}} \]

where \( |y - x| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \)
2.2.4 Green’s function
Green's functions on half-space $\mathbb{R}^n_+ = \{ (x_1, \ldots, x_n) \text{ with } x_1 > 0 \}$.

$\partial \mathbb{R}^n_+ =$ hyperplane $= \{ y = (y_1, \ldots, y_{n-1}, 0) \}.$

Define $\overline{x} = (x_1, \ldots, x_{n-1}, -x_n)$, the reflection in $\partial \mathbb{R}^n_+$ of point $x = (x_1, \ldots, x_{n-1}, x_n)$.

If $y \in \partial \mathbb{R}^n_+$, then

$$|y - \overline{x}|^2 = (y_1 - x_1)^2 + \cdots + (y_{n-1} - x_{n-1})^2 + (0 + x_n)^2$$

But $|y - x|^2 = (y_1 - x_1)^2 + \cdots + (y_{n-1} - x_{n-1})^2 + (0 - x_n)^2$

so $|y - \overline{x}| = |y - x|$. And since $\Phi$ is radial, i.e. $\Phi(z) = \Phi(|z|)$

it follows that for $y \in \partial \mathbb{R}^n_+$, $\Phi(y - x) = \Phi(|y - \overline{x}|) = \Phi(|y - x|) = \Phi(y - x)$.

Guess $G(x, y) = \Phi(y - \overline{x})$.

Define $G(x, y) = \Phi(y - x) - \Phi(y - \overline{x})$. 


Consider \( \frac{\partial}{\partial y} g(x,y) \) for \( y \in \mathbb{R}^n_+ \):

the outward normal \( v \) on \( \partial \mathbb{R}^n_+ \) is \( v = (0, 0, \ldots, 0, -1) \)

Calculate \( D \varphi(y) = -\frac{1}{n \sigma(n)} \frac{y}{|y|^n} \), \( n = 2 \) and \( n \geq 3 \)

Claim: \[ -\frac{\partial}{\partial y^n} \left( \varphi(y-x) - \varphi(y-x_n) \right) \bigg|_{y_n = 0} = \frac{2 \pi x_n}{n \sigma(n) |x-y|^n}, \quad x \in \mathbb{R}^n_+, \] called the Poisson kernel.
2.2.4 Green’s function
\[ u(x) = - \int \frac{\partial G(x, y)}{\partial x} \, ds(y) - \int G(x, y) \, \Delta u(y) \, dy \]

To solve: \[ \Delta u = f, \quad u = g \text{ on } \partial U, \]

Here, \( R^n_+ \) is unbounded, but recall for \( U \) bold suggests

\[ \begin{cases} \Delta u = 0 \quad \text{in } R^n_+ \\ \frac{\partial u}{\partial y} = g(y) \quad \text{on } \partial R^n_+ \end{cases} \]

has representation

\[ u(x) = - \int \frac{\partial G(x, y)}{\partial x} \, ds(y), \quad \text{where } G(x, y) = \Phi(y - \bar{x}) - \Phi(y - \bar{x}), \]

\[ \bar{x} = (x_1, x_2, \ldots, x_{n-1}, -x_n) \]

\[ \Phi^\pi(y) = \Phi(y - \bar{x}) \]
Calculate: \[ -\frac{2G(x,y)}{\alpha n} = \frac{2\alpha^n}{n\alpha(x)} \frac{1}{|x-y|^n} \quad x \in \mathbb{R}_+^n \]
\[ y \in \partial \mathbb{R}_+^n \quad \text{i.e.} \quad y_n = 0. \]

Then 14: \[ \forall \ g(y) \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1}) \]
\[ R^{n-1} = \partial \mathbb{R}_+^n. \]

and \( u(x) \) is given by Poisson's formula
\[ u(x) = \int_{\partial \mathbb{R}_+^n} K(x,y) g(y) \, ds(y) \quad \text{where} \ K \ \text{as above}, \]
\[ \partial \mathbb{R}_+^n \]
then:

\( \text{i) } u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}^{n-1}) \)

\( \text{ii) } \frac{\partial u}{\partial x} = 0 \quad \text{in} \ \mathbb{R}_+^n \)

\( \text{iii) } \lim_{x \to x_0} u(x) = g(x_0), \ \text{each} \ x_0 \in \partial \mathbb{R}_+^n \)
\[ x \in \mathbb{R}_+^n \]
(Partial) proof

1. For \( x, y \in \mathbb{R}_+^n \) and \( y \neq x \), then \( y \neq \bar{x} \) and \( G(x, y) \) is harmonic with respect to \( y \): \( \Delta_y G(x, y) = 0 \). But \( G(x, y) = G(y, x) \). \( \Rightarrow \Delta_x G(x, y) = 0 \), all pairs

\[ K(x, y) = -\frac{\partial G(x, y)}{\partial y} = \frac{\partial G(y, x)}{\partial y} \quad \text{as harmonic} \]

\( K(x, y) \) is harmonic in \( \mathbb{R}_+^n \) for \( x, y \in \mathbb{R}_+^n \), \( x \neq y \).

2. \( \int K(x, y) \, dS(y) \) \( \Rightarrow \)

\[ = \frac{2\pi n}{\alpha(n)} \int \frac{1}{\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 + (x_n - 0)^2}} \, dy_1 \cdots dy_{n-1} \]

Let \( z = (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n) \) \( \Rightarrow \) \( z \in \mathbb{R}_+^{n-1} \) is fixed.

\[ \int K(x, y) \, dS(y) = \frac{2\pi n}{\alpha(n)} \int \frac{1}{\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2 + x_n^2}} \, dz_1 \cdots dz_n \]

Let \( w = (z_1/x_n, z_2/x_n, \ldots, z_n/x_n) \) ; \( z_1 = x_n w_1, \ldots, z_n = x_n w_n \)

\[ \int K(x, y) \, dS(y) = \frac{2\pi n}{\alpha(n)} \frac{1}{\alpha(n)} \int \frac{1}{\sqrt{w_1^2 + w_2^2 + \cdots + w_n^2 + 1}} \, dw_1 \cdots dw_{n-1} (\text{etc.}) = 1. \]
3. If $g$ is odd, so is $u$:

$$u(x) = \int_{\partial \mathbb{R}^n_+} K(x, y) g(y) \, ds(y)$$

$$|u(x)| \leq \int_{\partial \mathbb{R}^n_+} |K(x, y)||g(y)| \, ds(y) \leq \|g\|_{L^\infty(\mathbb{R}^n_+)} \int_{\partial \mathbb{R}^n_+} K(x, y) \, ds(y)$$

Since $K(x, y) \in C^\infty$, it follows (after work)

also $u(x) \in C^\infty(\mathbb{R}^n_+)$.

$$\Delta_x u(x) = \int_{\partial \mathbb{R}^n_+} \Delta_x K(x, y) g(y) \, dy = \delta^n 0 \quad x \in \mathbb{R}^n_+$$

(some work to show)

(interchange o.k.)

(it follows $\Delta_x K(x, y) = 0$.)

$$|K| = K$$
Next, fix $x^0 \in \partial \Omega^\alpha$. We will show that $\lim_{x \to x^0} u(x) = g(x^0)$.

![Diagram of a 3D coordinate system with axes labeled $x_1$, $y_1$, $x_2$, $y_n$, and point $x_0$.]

$$|u(x) - g(x^0)| = 1 \int_{\partial \Omega^\alpha} K(x, y)(g(y) - g(x^0)) \, dy \leq \int_{\partial \Omega^\alpha \cap B(x^0, \delta)} K(x, y)|g(y) - g(x^0)| \, dy$$

$$+ \int_{\partial \Omega^\alpha - B(x^0, \delta)} K(x, y)|g(y) - g(x^0)| \, dy.$$
2.2.4 Green’s function
Consider $I : y \in \mathbb{R}_+^n \cap B(x^0, \delta) \Rightarrow |y - x^0| < \delta$.

For any $\varepsilon > 0$, there is a $\delta > 0$ such that $|y - x^0| < \delta \Rightarrow |g(y) - g(x^0)| < \varepsilon$. 

\[
|u(x) - g(x^0)| = \left| \int_{\partial \mathbb{R}_+^n} K(x, y) (g(y) - g(x^0)) \, dy \right|
\]

\[
\leq \int_{\partial \mathbb{R}_+^n} K(x, y) |g(y) - g(x^0)| \, dy
\]

\[
\leq \int_{\partial \mathbb{R}_+^n \cap B(x^0, \delta)} |g(y) - g(x^0)| \, dy + \int_{\partial \mathbb{R}_+^n \setminus B(x^0, \delta)} |g(y) - g(x^0)| \, dy
\]
then \( I \leq \int_{\partial \mathbb{R}^n_+ \cap B(\kappa^0, \delta)} K_0(x, y) \, dy \leq \int_{\partial \mathbb{R}^n_+} K_0(x, y) \, dy = \varepsilon. \)

Consider \( J : \ y \in \partial \mathbb{R}^n_+ - B(\kappa^0, \delta) \)

Suppose \( \kappa \in B(\kappa^0, \frac{\delta}{2}) \), then

\[
|y - \kappa^0| \leq |y - \kappa + \kappa - \kappa^0| \\
\leq |y - \kappa| + |\kappa - \kappa^0| \\
\leq |y - \kappa| + \frac{\delta}{2} \\
\leq |y - \kappa| + \frac{1}{2} |y - \kappa^0| \]

\[
\Rightarrow \frac{1}{2} |y - \kappa^0| \leq |y - \kappa|.
\]
\[ J = \int_{\partial \mathbb{R}^n_+ - B(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dy \]

\[ \leq 2 \|g\|_\infty \int_{\partial \mathbb{R}^n_+ - B(x^0, \delta)} \frac{x_n}{n \alpha(n)} \frac{1}{(y-x)_{\alpha(n)}} dy \]

\[ \leq 2 \|g\|_\infty \int_{\partial \mathbb{R}^n_+ - B(x^0, \delta)} \frac{x_n}{n \alpha(n)} \left( \frac{1}{\frac{1}{2} |y-x^0|} \right)^n dy \]

\[ = 2^{n+2} \|g\|_\infty \frac{x_n}{n \alpha(n)} \int_{\partial \mathbb{R}^n_+ - B(x^0, \delta)} \left( |y-x^0|^{-n} \right) dy \]

\[ \text{bounded} \]

\[ \Rightarrow J \to 0 \text{ as } x_n \to 0. \]

\[ \Rightarrow \lim_{x \to x^0} u(x) = g(x^0), \quad x^0 \in \partial \mathbb{R}^n_+. \]
2.2.4 Green’s function
Green's function for $B(0, 1)$

want: $G(x, y) = \Phi(y-x) - \Phi^*(y)$, where $\Phi^*(y)$ is harmonic

\[
\Delta y \Phi(y) = 0
\]

and $\Phi^*(y) = \Phi(y-x)$ on $\partial B(0, 1)$.

at $y \in \partial B(0, 1)$, $\Phi(y-x) = \Phi(l_y-y_1)$, and

\[
\|y-x\|^2 = (y-x) \cdot (y-x)
\]

\[
= y_1^2 - 2y \cdot x + x_1^2
\]

\[
= 1 - 2y \cdot x + x_1^2
\]

Let $\mathbf{x}$ be the point $\mathbf{x} = \frac{x}{\|x\|^2}$.

then $\mathbf{x} \cdot x = 1$; $\mathbf{x}$ along same direction as $x$, but outside $B(0, 1)$. 
Consider $1y - \hat{x}1$ for $y \in \partial B(0, 1)$

\[1y - \hat{x}1^2 = 1y1^2 - 2y \cdot \hat{x} + 1\hat{x}1^2 = 1 - 2y \cdot \frac{x}{1x1^2} + \frac{1x1^2}{1x1^2}
\]

\[= \frac{1}{1x1^2} \left( 1x1^2 - 2y \cdot \hat{x} + 1 \right)
\]

\[1y - \hat{x}1^2 = \frac{1}{1x1^2} (1 - 2y \cdot \hat{x} + 1x1^2)
\]

Compare with $1y - x1^2$ from above. Find

\[1y - \hat{x}1^2 = \frac{1}{1x1^2} 1y - x1^2
\]

\[\text{w} : 1x11y - \hat{x}1 = 1y - x1, \ \text{true for } y \in \partial B(0, 1)
\]

$\Phi(y)$ is radial: $\Phi(y) = \Phi(1y1)$.

\[\therefore \Phi(1x1(y - \hat{x})) = \Phi(1x11y - \hat{x}1) = \Phi(1y - x1) = \Phi(y - x)
\]

Define: $\phi^x(y) = \Phi(1x1(y - \hat{x}))$.

Then: $\phi^x(y)$ is continuous, $y \in B(0, 1)$; also $C^\infty(B(0, 1))$; also $\Delta_y \Phi(1x1(y - \hat{x})) = 0$; also analytic
Define Green's function for the unit ball $B(0,1)$ as:

$$G(x, y) := \frac{1}{2\pi} \log |x-y|, \quad x, y \in B(0,1), \quad x \neq y.$$ 

Qf: $u \in C^2(B(0,1))$ and:

$$\Delta u = 0 \text{ in } B^0(0,1)$$
$$u = g \text{ on } \partial B(0,1),$$

then (claim)

$$u = -\int_{\partial B(0,1)} g(y) \frac{\partial}{\partial y} G(x, y) \, dS(y).$$
2.3.3 Properties of solutions
Recall

\[
\frac{u^e(x + he_i) - u^e(x)}{h} = \int_{\Omega} \frac{\mathcal{N}_e(x + he_i, y) - \mathcal{N}_e(x, y)}{h} u(y) \, dy.
\]

Take \( h < \epsilon \); \( \mathcal{N}_e(x + he_i, y) \subset B^0(x + he_i, \epsilon) \subset B^0(x, 2\epsilon) \subset \Omega \)

\[
B(x, 2\epsilon) \subset \Omega
\]

\[
\lim_{h \to 0} \frac{\mathcal{N}_e(x + he_i, y) - \mathcal{N}_e(x, y)}{h} = \frac{\partial \mathcal{N}_e(x, y)}{\partial x_i}
\]

uniformly for \( y \in B(x, 2\epsilon) \)

\( u^e \) is differentiable w.r.t. \( x_i \)

and

\[
\frac{\partial u^e(x)}{\partial x_i} = \int_{B(x, 2\epsilon)} \frac{\partial \mathcal{N}_e(x, y)}{\partial x_i} u(y) \, dy
\]
In addition, (claim) \( \frac{\partial u}{\partial x} (x) \) is continuous as a function of \( x \),

because \( \frac{\partial u}{\partial x_1} (x, y) \) is continuous on \( B(x, 2\varepsilon) \)

... uniformly continuous.

See Appendix C Theorem 7 for \( D^a u, \quad a = (x_1, x_2, \ldots, x_n), \)

any order \( 1 \times 1 \).

Towards analyticality: Estimates on derivatives.

If \( |x| = k \) and \( B(x, k) \subset U \), \( u \) harmonic on \( U \),

then

\[ |D^a u(x_0)| \leq \frac{C_k}{n! k^a} \|u\|_{L^1(B(x, k))}, \]

\[ C_k = \frac{1}{\kappa(k)}, \]

\[ C_k = \frac{(2^{n+1} + n)^k}{\alpha(n)}, \]

\( k \geq 1 \).
\[ \text{For } k = 0 \quad D^{(0,1,\ldots,0)} u(x_0) \text{ means just } u(x_0) \]

\[ |u(x_0)| = \left| \int_{B(x_0, \gamma)} u(y) \, dy \right| \leq \int_{B(x_0, \gamma)} |u(y)| \, dy \]

\[ = \left( \frac{1}{\text{vol. of } B(x_0, \gamma)} \right) \cdot \int_{B(x_0, \gamma)} |u(y)| \, dy = \frac{1}{\alpha(n) \gamma^n} \| u \|_{L^1(B(x_0, \gamma))} \]

\[ \leq \frac{C_0}{\gamma^n} \| u \|_{L^1(B(x_0, \gamma))} \quad \text{as desired, for } k = 0. \]

For \( k = 1 \)

\[ u_{\kappa_1}(x) = \int_{B(x_0, \gamma/2)} u_{\kappa_1}(y) \, dy \quad \text{from thin 6, } \]

\[ u \in C^\infty \text{ so this is o.k.} \]

\[ = \frac{1}{\text{vol. of } B(x_0, \gamma/2)} \int_{B(x_0, \gamma/2)} u_{\kappa_1}(y) \, dy. \]
\[ u_{\kappa} (x_0) = \frac{1}{\text{vol. of } B(x_0, \frac{\kappa}{2})} \int_{\partial B(x_0, \frac{\k}{2})} u \nu^i ds(y), \quad \nu = (\nu^1, \ldots, \nu^m) \text{ is unit outward normal at } y \in \partial B(x_0, \frac{\k}{2}) \]

(recall \[ \int_{\Omega} u \nu \cdot d\mathbf{x} = \int_{\Omega} u \nu^i ds(x) - \int_{\partial B(x_0, \frac{\k}{2})} u \nu^i \cdot d\mathbf{x} \]

\[ u = 1 ; \quad \nu \cdot \nu = 0 \] gives

\[ \int_{\Omega} u \nu_i \cdot d\mathbf{x} = \int_{\partial B(x_0, \frac{\k}{2})} u \nu^i ds(x) \] \]

\[ |u_{\kappa} (x_0)| \leq \frac{1}{\alpha(n) \left( \frac{\k}{2} \right)^n} \cdot \alpha(n) \left( \frac{\k}{2} \right)^{n-1}  ||u||_{L^\infty (\partial B(x_0, \frac{\k}{2}))} \cdot 1 \]

\[ = \frac{2m}{n} ||u||_{L^\infty (\partial B(x_0, \frac{\k}{2}))} \]

Bound for \( ||u||_{L^\infty (\partial B(x_0, \frac{\k}{2}))} \) :

For each \( y \in \partial B(x_0, \frac{\k}{2}) \),

\[ u(y) = \int_{B(y, \frac{\k}{2})} u(z) dz \]

\[ = \frac{1}{\text{vol. of } B(y, \frac{\k}{2})} \int_{B(y, \frac{\k}{2})} u(z) dz \]
So: \( |u(y)| \leq \frac{1}{\text{vol } B(y, \frac{\nu}{2})} \|u\|_1(B(y, \nu)) \).

\[
\|u\|_{L^\infty}(\partial B(x, \frac{\nu}{2})) = \max_{y \in \partial B(x, \frac{\nu}{2})} |u(y)| \\
\leq \frac{1}{\text{vol } B(y, \frac{\nu}{2})} \|u\|_{L^2}(B(y, \frac{\nu}{2})).
\]

\[
|u_{x_1}(x_0)| \leq \frac{2^n}{\nu} \|u\|_{L^\infty}(\partial B(x_0, \frac{\nu}{2})) \\
\leq \frac{2^n}{\nu} \|u\|_{L^\infty}(B(y, \frac{\nu}{2})) \\
\leq \frac{2^{n+1} n}{\nu^{n+1} \alpha(n)} \|u\|_{L^1(B(x_0, \nu))}.
\]

For \( k = (\alpha_1, \ldots, \alpha_n) \), \( |D^\alpha u(x_0)| = |u_{x_1}(x_0)| \), where \( \alpha = (0, 0, \ldots, 1, 0, \ldots, 0) \)\

\[
|D^\alpha u(x_0)| \leq \frac{C_1}{\nu^{n+1}} \|u\|_1(B(x_0, \nu)), \quad C_1 = \frac{2^{n+1} n}{\alpha(n)}.
\]

as claimed for \( k = 1 \).