3.4 Introduction to conservation laws
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§3.4. Intro. to Conservation Laws

Given $F: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$,

consider

\[ u_t + (F(u))_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \]

\[ u = g \quad \text{for} \quad \mathbb{R} \times \{t = 0\} \]

From study of characteristics,
this problem may not have solutions with enough continuity or differentiability, to substitute into the PDE.

How does one interpret $u$ that does not have the required derivatives, as a "weak" or "generalized" solution?
Idea: take smooth function \( v \), with compact support, call \( v \) "test function".

Take \((PDE) \times v\) and integrate

\[
\int_{t>0} \int_{x \in \mathbb{R}} [u_t + (F(u))_x] v \, dx \, dt
\]

Integrate by parts:

Consider

\[
\int_{t>0} u_t v \, dt = \left. u v \right|_{t=0}^{t=\infty} - \int_{t>0} u v_t \, dt
\]

\[
= -u(x,0)v(x,0) - \int_{t>0} u v_t \, dt
\]

So:

\[
\int_{t>0} \int_{x \in \mathbb{R}} u_t v \, dx \, dt = -\int_{x \in \mathbb{R}} u(x,0)v(x,0) \, dx - \int_{t>0} \int_{x \in \mathbb{R}} u v_t \, dt - \int \int_{t>0} \int_{x \in \mathbb{R}} u v_t \, dx \, dt.
\]

(assume interchange order o.k.)
Consider
\[ \int_{\mathbb{R}} (F(u))_x u \, dx = F(u)u \bigg|_{-\infty}^{\infty} - \int_{\mathbb{R}} F(u) u_x \, dx \]

since \( u \) has compact support.

\[ \int_{t>0} \int_{\mathbb{R}} (F(u))_x u \, dx \, dt = -\int_{t>0} \int_{\mathbb{R}} F(u) u_x \, dx \, dt \]

From PDE, find
\[ 0 = \int_{t>0} \int_{\mathbb{R}} \left[ u u_t + (F(u))_x \right] \, dx \, dt \]

(4) \[ 0 = -\int_{t>0} \int_{\mathbb{R}} (u u_t + F(u) u_x) \, dx \, dt - \int_{\mathbb{R}} \frac{u(x,v) u(x,0)}{g(x)} \, dx \]

Define \( u \in L^\infty(\mathbb{R} \times (0,\infty)) \) as an integral solution of (1),
provided (4) holds for each test function \( v \).
If we have an integral solution, what does (4) allow us to do with u? (say about u).

Suppose a region V, divided into left and right parts \( V_L \) and \( V_R \).

and \( u \) is an integral solution, such that: \( u \) and its first deriv. uniformly continues in each of \( V_L \) and \( V_R \).