3.4.2 Lax-Oleinik formula
"Review" Legendre transform

Suppose \( L : \mathbb{R}^n \to \mathbb{R} \) obeys \( q \mapsto L(q) \) convex.

\[
\text{and } \lim_{|q| \to \infty} \frac{L(q)}{|q|} = +\infty.
\]

\[
\text{ex/ } F(q) : \mathbb{R}^n \to \mathbb{R} \text{ given by } F(q) = \frac{q^2}{2}.
\]

\[
\frac{q^2}{2|q|} = \frac{1}{2} \to \infty \text{ as } |q| \to \infty.
\]

Define \( L^*(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \} \), \( p \in \mathbb{R}^n \).

\[
\text{ex/ } F^*(p) = \sup_{q \in \mathbb{R}^1} \{ pq - F(q) \}.
\]

\[
= \max_{q \in \mathbb{R}} \{ pq - \frac{q^2}{2} \} = \max_{q \in \mathbb{R}} \left\{ \frac{1}{2} (2pq - q^2) \right\} \]

\[
= \frac{1}{2} \max_{q} \left\{ -(q-p)^2 + p^2 \right\} \text{ complete the square}
\]

\[
= \frac{1}{2} \left( -(q-p)^2 + p^2 \right) = \frac{p^2}{2}.
\]
Another way: Consider \( \max_q \{ pq - F(q) \}^3 \).

If occurs, then \( \frac{1}{q} \frac{d}{dq} (pq - F(q)) = 0 \) (or \( \frac{1}{q} ; p \) fixed)

\[ p - F'(q) = 0 \]

Suppose \( G = (F')^{-1} \); then \( q = G(p) \), at maximum so at maximum,

\[ F^*(p) = \int pq - F(q)^3 = pG(p) - F(G(p)). \]

Check: for \( F(q) = q^{3/2} \), \( F'(q) = q \)

\[ G(p) = p \text{ is inverse} \]

\[ F^*(p) = p \cdot p - F(p) \]
\[ = p^2 - p^{3/2} = p^{3/2}. \]
53.4.2: Towards a single formula for weak solution of IVP

\[ ut + (F(u))_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \]

\[ u(x, 0) = g(x) \text{ at } t = 0. \]

Assume: \( F'' \geq 0 > 0 \), so \( F \) is uniformly convex
Also assume \( F(0) = 0 \). without loss of generality.

In 53.3, the Hamilton–Jacobi equation

\[ wt + F(w_x) = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \]

\[ w(x, 0) = h(x) \]

is solved: the soln is

\[ w(x, t) = \min_{y \in \mathbb{R}} \left\{ t + L\left( \frac{x-y}{t} \right) + h(y) \right\} \]

(The unique weak soln)

where \( L(q) = \max_{q \in \mathbb{R}} \left\{ pq - F(q) \right\} \)
where "weak solution" means:

a) \( w(x, v) = A(x) \),

b) \( w_t + F(w_x) = 0 \) a.e. \((x, t) \in \mathbb{R} \times (0, \infty)\)

c) \( w(x+3, t) - 2w(x, t) + w(x-3, t) \leq C(1 + \frac{1}{t})18^2 \)

for some \( C > 0 \), \( x, y \in \mathbb{R} \) and \( t > 0 \).

Here, for

\[
\begin{cases}
    w_t + (F(w))_x = 0 \\
    w(x, 0) = g(x)
\end{cases}
\]

let \( A(x) = \int_0^x g(y) \, dy \); consider b), differentiate w.r.t \( x \):

\( w_{xt} + (F(w_x))_x = 0 \)

Similarly, differentiate a):

\( w_x(x, 0) = A'(x) = g(x) \)
Expect $u = u_x$ should satisfy Burger's PDE.

But $w(x, t) = \min_{y \in \mathbb{R}} \left\{ t \wedge \left( \frac{x-y}{t} \right) + k(y) \right\}$.

So: expect

$$ u(x, t) = \frac{2}{\partial x} \left[ \min_{y \in \mathbb{R}} \left\{ t \wedge \left( \frac{x-y}{t} \right) + k(y) \right\} \right] $$

So will be defined a.e. $(x, t)$, and (should be) candidate for a weak soln of IVP (1).

Thin (Lax-Oleinik formula)