3.2.2 Examples
Example: linear

\[ \nabla \cdot (\kappa \nabla u) + \kappa_2 u_{\kappa_2} = u \quad \text{in} \]

PDE of form,

\[ F(Du, u, x) = 0 \]

where

\[ F(p, g, x) = \mathbf{b}(x) \cdot p + cg, \]

and \( \mathbf{b}(x) = (-\kappa_2, \kappa_1) \), \( c = -1 \).

Could use procedure from last day

Another approach:

Suppose can find \( z(s) = u(x_1(s), x_2(s)) \).

Want:

\[ \frac{\partial z}{\partial s} = \frac{\partial u}{\partial x_1}(x_1(s)) + \frac{\partial u}{\partial x_2}(x_2(s)) \]

Compare with \( \frac{\partial u}{\partial \kappa_1} + \frac{\partial u}{\partial \kappa_2} + \kappa_2 u_{\kappa_2} = u \)
We choose to make a) \( \dot{x}_1(s) = -x_2, \dot{x}_2(s) = x_1 \).

Then also b) \( \dot{z}(s) = z(s) \).

If \( x(0) \in \Gamma \), then

\[ x(0) = (x^0, 0) \]

and solution of a) is

\[ x(s) = (x^0 \cos s, x^0 \sin s) \]

Also \( z(s) = z(0)e^s \); and at \( s = 0 \),

\[ z(0) = u(x_1(0), x_2(0)) = u(x^0, 0) = g(x^0) \]

\[ \therefore z(s) = g(x^0)e^s \]

Now to find, for general \( x \in \Gamma \), the soln \( u(x) \):

Write \( (x_1, x_2) = r \cos \theta \),

Then the (projected) characteristic with \( x^0 = r \),

intersects the point \( (x_1, x_2) \) for \( s = 0 \).

\[ r = x^0 \]

and \( u(x_1, x_2) = z(0) = g(0)e^0 = g(x^0) \).

\[ \theta = \arctan(x_2, x_1) \]
Consider quasi-linear

\[ F(Du, u, x) = 0 \]

where

\[ F(\rho, z, x) = b(x, z) \cdot \rho + c(x, z) \]

Then

\[ D_\rho F = b(x, z) \]

(11c) \[ \dot{x} = D_\rho F(\rho, z, x) = b(x, z) \]

(11d) \[ \dot{z} = D_\rho F \cdot \rho = b(x, z) \cdot \rho. \]

For \( Du = \rho(s), u = z(s), x = x(s), \)

PDE along \( \rho(s), z(s), x(s) \) is

\[ b(x(s), z(s), \rho(s)) \cdot \rho(s) + c(x(s), z(s)) = 0 \]

(11e) \[ \dot{z} = -c(x(s), \rho(s)). \]
\[ u_{x_1} + u_{x_2} = u^2 \]

is of form,
\[ F(Du, u, x) = 0 \]

where
\[ F(p_1, p_2, z, x_1, x_2) = p_1 + p_2 - z^2 \]
\[ = (1, 1) \cdot p - z^2 \]
\[ = b \cdot p - z^2 \]
\[ c \text{ is nonlinear.} \]

Equations 17.
\[
\left\{ \begin{array}{l}
\dot{x}_1 = 1 \\
\dot{x}_2 = 1 \\
\dot{z} = b \cdot p = z^2, \text{ since } F = 0 \text{ along } (p(t), z(t), x(t))
\end{array} \right.
\]
For initial \( x(0) = (x_1(0), x_2(0)) \) on \( \Gamma \),
\[
x(0) = (x^0, 0)
\]
\[
\dot{x}_1 = 1 \Rightarrow x_1(s) = x^0 + s
\]
\[
\dot{x}_2 = 1 \Rightarrow x_2(s) = 0 + s
\]
\[
\dot{z} = z^2; \quad \frac{\dot{z}}{z^2} = \frac{d}{ds} \left( -\frac{1}{z} \right) = 1 \Rightarrow -\frac{1}{z(s)} = -\frac{1}{z(0)} + s,
\]
where
\[
z(s) = \frac{g(x^0)}{1 - s g(x^0)}
\]

Aside
\[
\begin{align*}
\frac{1}{z(s)} &= \frac{1}{z(0)} - s \\
3(s) &= \frac{1}{\frac{1}{z(0)} - s} = \frac{z(0)}{1 - s z(0)} = \frac{g(x^0)}{1 - s g(x^0)}
\end{align*}
\]

If \( x = (x_1, x_2) \in U \), solve \( \int (x^0 + s = x_1) \) for \( s \) and \( x_0 \)
\[
\begin{align*}
\int_0^s \frac{1}{z_2(s)} ds &= x_1 - x_2 \\
\text{find: } s &= x_2, x_0 = x_1 - x_2
\end{align*}
\]

Solve \( u(x_1, x_2) = z(x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)} \).