2.3 Heat equation
Heat equation $\S 2.3.1.$

A derivation

Let $u(x, t)$ be

density of some quantity $\leftrightarrow$ concentration of salt in gm/cm$^3$

$\leftrightarrow$ amount of heat energy in joules/cm$^3$

in $U$.

Let $V \subset U$, then the amount of $u$ in $V$

is $\int_V u(x, t) \, dx$.

The rate of change of this amount, is

\[ \frac{\partial}{\partial t} \int_V u(x, t) \, dx \]
Suppose there is a flow (flux) of the quantity, defined by a flux vector \( \vec{F} \) such that: the amount passing per unit time through an element of surface \( \Delta S \) with normal \( \vec{n} \) (unit), is \( (\vec{F} \cdot \vec{n}) \Delta S \).

Then the amount passing through the surface of \( V \) per unit time is

\[
\int_V \left( \vec{F} \cdot \vec{n} \right) dS(y) \\
y \in dV
\]

If \( \vec{n} \) is outward, this integral gives rate at which quantity leaves \( V \):

\[
\frac{\partial}{\partial t} \int_V u(x,t) dS = - \int_V (\vec{F} \cdot \vec{n}) dS(y). 
\]
Suppose \( \hat{F} \in C'(U) \).

By divergence theorem,
\[
\int_V (\hat{F} \cdot \nu) \, ds(y) = \int_V \text{div} \hat{F} \, dx
\]
so (assuming \( V \) has fixed, time-independent boundary)
\[
\int_V \left( \frac{\partial u}{\partial t}(x,t) + \text{div} \hat{F} \right) \, dx = 0.
\]

\( V \) was arbitrary; if \( \frac{\partial u}{\partial t} \) and \( \text{div} \hat{F} \) are continuous,
\[
\frac{\partial u}{\partial t}(x,t) + \text{div} \hat{F} = 0 \quad \text{in} \ U.
\]

Qf: \( \hat{F} = \nabla u \), this gives
\[
\frac{\partial u}{\partial t} = \Delta u \text{ gradient } \nabla
\]
\[
\frac{\partial u}{\partial t} = a \Delta u : \text{ diffusion again.}
\]
\[
\frac{\partial u}{\partial t} = a \nabla^2 u : \text{ heat again if } u = \text{ energy density.}
\]