Problem 2.5.6

Let \( y = U + \frac{1}{2n} \lambda \), where \( \lambda = \max \frac{1}{u} \).

Then, for any \( i \in \mathbb{N} \),
\[
\frac{\partial y}{\partial x_i} = \frac{\partial U}{\partial x_i} + \frac{1}{2n} \lambda,
\]
where \( U(t, \ldots, x_i) \).

\[
\frac{\partial y}{\partial \lambda} = \frac{\partial U}{\partial \lambda} + \frac{1}{2n} \lambda.
\]

Similarly, \( \Delta U = -\delta \) in \( U \), we have
\[
\Delta U = \Delta U + \lambda = -\delta + \lambda > 0, \quad \lambda = \max \frac{1}{u}.
\]

Thus, we obtain \( \Delta U \) is subharmonic.

From problem 2.5.5 we, we obtained
\[
\max \frac{U}{\lambda} = \max \frac{V}{\lambda}.
\]

Note that \( U = \frac{V}{\lambda} \) on \( dU \), therefore
\[
\max \frac{U}{\lambda} + \frac{1}{2n} \lambda = \max \frac{V}{\lambda} + \frac{1}{2n} \lambda.
\]

Therefore, for any \( a \in U \),
\[
U(a) \leq U(a) + \frac{1}{2n} \lambda = \max \frac{V}{\lambda} + \frac{1}{2n} \lambda \leq \max \frac{V}{\lambda} + \frac{1}{2n} \lambda \leq \max \frac{V}{\lambda} + \frac{1}{2n} \lambda \leq \lambda
\]
for all \( a \in U \).

Therefore, \( U(a) \leq \lambda V \) for all \( a \in U \).

Similarly, set \( w = -U + \frac{1}{2n} \lambda \). It could be proved that \( w \) is also subharmonic. Then applying problem 2.5.5 again gives us
\[
\max \frac{w}{\lambda} = \max \frac{w}{\lambda}.
\]

By the similar argument as we had before, we have
\[
-U(a) \leq \max \frac{V}{\lambda} - \frac{1}{2n} \lambda + MA, \quad V a i U.
\]

Therefore, \( \max \frac{|U|}{\lambda} \leq \max \frac{|V|}{\lambda} + MA \lambda \)
\[
= \max \frac{|V|}{\lambda} + M \max \frac{1}{u} \lambda
\]

Take \( C = \max \frac{1}{u}, \lambda \), we have
\[
\max \frac{|U|}{\lambda} \leq C \left( \max \frac{|V|}{\lambda} + \max \frac{1}{u} \right),
\]
where \( C \) is depending only on \( U \).