MATH 649
Midterm
Mar. 11, 2015

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

• Please try all questions. This is a closed-book exam. However, if there is any information you need, please ask and it will be written on the board. Please show all work. You may use back pages if necessary.

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1. (10 points) Suppose $u$ satisfies

$$-\Delta u + cu = 0 \text{ in } U \in \mathbb{R}^n, \text{ U open, bounded}$$

Here $c$ is a constant. Fix $x \in U$ and define

$$\phi(r) = \frac{1}{\text{sfec area of } \partial B(x, r)} \int_{\partial B(x, r)} u(y) dS(y)$$

Find a differential equation for $\phi(r)$.

Hints: Volume of $B(x, r)$ is $\alpha(n)r^n$, surface area is $n\alpha(n)r^{n-1}$,

$B(x, r) = \{y \in \partial B(x, s), 0 \leq s < r\}$,

Green's formula $\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial v} \, dS$
\[ \phi(x) = \frac{1}{n \alpha(n) n^{n-1}} \int_{\partial B(x, r)} u(y) \, ds(y) \]
\[ = \frac{1}{n \alpha(n) n^{n-1}} \int_{\partial B(x, r)} u(x + n \zeta) \, n^{-1} \, ds(\zeta) \]
\[ = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} u(x + n \zeta) \, ds(\zeta), \quad (ds(\zeta))^{n-1} = ds(y) \]

\[ \phi'(x) = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} \frac{\partial}{\partial \nu} u(x + n \zeta) \, ds(\zeta) \]

\[ \frac{\partial}{\partial \nu} u(x + n \zeta) = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \frac{2}{\alpha(n)} (x_i + n \zeta_i) = \sum_{i=1}^{n} 3 \zeta \cdot \frac{\partial u}{\partial y_i} \]

\[ \phi''(x) = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} \frac{\partial u}{\partial y} (x + n \zeta) \, ds(\zeta) \]

\[ = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} \frac{\partial u}{\partial y} (y) \, \frac{1}{n} \, ds(y) \]

\[ = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} \frac{\partial u}{\partial y} (y) \, ds(y) \]

\[ = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} \Delta u \, ds(y) \]

\[ = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} c u(y) \, dy. \]

(\textit{extra}): \quad \zeta = (y - x)/r; \quad dy = n^{-1} \, ds

\[ \phi''(x) = \frac{1}{n \alpha(n)} \int_{\partial B(x, r)} c u(x + n \zeta) \, n^{-1} \, ds(\zeta) \]
But $B(x, r) = \{ y \in DB(x, r), 0 \leq s \leq r \}$

$$\int_{B(x, r)} u(y) \, dy = \int_{0}^{r} \left[ \int_{DB(x, s)} u(y) \, dS(y) \right] \, ds$$

Now $\phi(s) = \frac{1}{n \alpha(n) s^{n-1}} \int_{DB(x, s)} u(y) \, dS(y)$, so

$$\int_{B(x, r)} u(y) \, dy = \int_{0}^{r} n \alpha(n) s^{n-1} \phi(s) \, ds.$$ 

It follows that

$$\phi(r) = \frac{1}{n \alpha(n) r^{n-1}} \int_{B(x, r)} u(y) \, dy$$

$$= \frac{1}{n \alpha(n) r^{n-1}} \cdot \left[ \int_{0}^{r} n \alpha(n) s^{n-1} \phi(s) \, ds \right]$$

$$= \frac{r^{n-1}}{n} \int_{0}^{r} s^{n-1} \phi(s) \, ds.$$ 

To find a DE for $\phi(r)$, take $d$:

$$d \left( r^{n-1} \phi' \right) = c \int_{0}^{r} s^{n-1} \phi(s) \, ds$$

$$(r^{n-1} \phi')' = c n^{n-1} \phi(r)$$

$$r^{n-1} \phi'' + (n-1) r^{n-2} \phi' - c n^{n-1} \phi(r) = 0$$
2. (10 points) Consider

\[-\Delta u = f \quad \text{in } U \subset \mathbb{R}^n, \ U \text{ open, bounded, connected}\]

\[u + \frac{\partial u}{\partial n} = g \quad \text{on } \partial U\]

where \(f \in C^2(\overline{U})\) and \(g \in C^2(\partial U)\).

Use the maximum principle to show there is at most one solution.

Let \(u_1, u_2\) be solutions and let \(w = u_1 - u_2\).

Then \(\Delta w = 0\) in \(U\) and \(w + \frac{\partial w}{\partial n} = 0\) on \(\partial U\)

where \(n\) is outward normal. Let \(\nu = -n\) be inward.

Then on \(\partial U\), \(\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial \nu} = 0\).

Suppose \(w\) attains max. value at \(x_M \in \partial U\).

Then \(\frac{\partial w}{\partial \nu}(x_M) = \lim_{h \to 0^+} \frac{w(x_M + h\nu) - w(x_M)}{h} = w(x_M)\).

If \(w(x_M) > 0\), for all suff. small \(h > 0\), \(w(x_M + h\nu) - w(x_M) > 0\)

so there are points \(x = x_M + h\nu \in U\) where \(w(x) > w(x_M)\),

a contradiction. \(\therefore w(x) \leq 0\).

Suppose \(w\) attains its min. value at \(x_m \in \partial U\).

Then \(\frac{\partial w}{\partial \nu}(x_m) = \lim_{h \to 0^+} \frac{w(x_m + h\nu) - w(x_m)}{h} = w(x_m)\).

If \(w(x_m) < 0\), then for all suff. small \(h > 0\), \(w(x_m + h\nu) - w(x_m) < 0\)

so there are points \(x = x_m + h\nu \in U\) where \(w(x) < w(x_m)\),

a contradiction \(\therefore w(x_m) \geq 0\).

Since \(\min_U w = w(x_m) \geq 0\) and \(\max_U w = w(x_M) \leq 0\),

\(\min_U w \leq w(x)^3 \leq \max_U w \Rightarrow w(x) = 0 \quad \text{for all } x \in U\).
3. (10 points) Consider

\[-\Delta u = 0 \text{ in } \mathbb{R}^n - B(0,1), \ n \geq 3\]

\[u = g \text{ on } \partial B(0,1)\]

Perform the following:

a) write down a formal representation of the solution in terms of the Green's function \(G(x, y)\),

b) simplify the representation by calculating \(\frac{\partial G}{\partial n}(x, y) = \nu \cdot D_y G(x, y), \ y \in \partial B(0,1)\).

Hints: Recall that for \(U\) bounded, \(u \in C^2(\bar{U})\)

\[u(x) = -\int_{\partial U} u(y) \frac{\partial G}{\partial n}(x, y) dS(y) - \int_U G(x, y) \Delta_y u(y) dy\]

where \(G(x, y) = \Phi(y - x) - \Phi^*(y)\) has the property, \(G(x, y) = 0\) for \(x \in U, \ y \in \partial U\). Although here \(\mathbb{R}^n - B(0,1)\) is unbounded the given formula should inspire your representation. You may assume without proof,

1. \(\Phi(y) = \frac{1}{n(n-2)\alpha(n)|y|^{n-2}}\), \(D_y \Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n}\), \(\Delta_y \Phi(y) = 0\) for \(y \neq 0\)

2. If \(|x| > 1\) and \(\tilde{x} = \frac{x}{|x|^2}\), then \(|x||y - \tilde{x}| = |y - x|\) for all \(y \in \partial B(0,1)\)

3. If \(\phi^*(y) = \Phi(|x|(|y - \tilde{x})|), \) then \(\Delta_y \phi^*(y) = 0\) for \(|y| > 1\)

\(a)\) \(u(x) = -\int_{\partial B(0,1)} \Phi(y) \frac{\partial G}{\partial n}(x, y) dS(y) \quad \text{since} \quad \Delta u(y) = 0 \text{ in } \mathbb{R}^n - B(0,1)\)

\(b)\) \(\frac{\partial G}{\partial n}(x, y) = \nu \cdot D_y G(x, y) \quad \nu = \frac{y}{|y|} = -\frac{y}{|y|}\)

\[G(x, y) = \Phi(y - x) - \Phi^*(y)\]

\[D_y G(x, y) = D_y \Phi(y - x) - D_y \Phi^*(y) \quad \text{for } y \in \partial B(0,1)\]

\[= \frac{-1}{n\alpha(n)} \frac{y-x}{|y-x|^n} - \frac{-1}{n\alpha(n)} \frac{|x|}{|y-x|^n} \frac{|x|}{|y-x|^n}\]

\[= \frac{1}{n\alpha(n)} \frac{|y-x|^n}{|y-x|^n} \left( \frac{|x|^n}{|y-x|^n} - \frac{|x|^n}{|y-x|^n} \right) \]

\[= \frac{1}{n\alpha(n)} \frac{|y-x|^n}{|y-x|^n} \left( (|x|^2 - 1)y - 1|y|^2 + x \right) \]

\[= \frac{1}{n\alpha(n)} \frac{|y-x|^n}{|y-x|^n} \left( (|x|^2 - 1)y - 1|y|^2 + x \right) \]

\[\text{would cancel these terms} \]

\[\text{mistakes on which real} \]
Then \( \frac{\partial G}{\partial y}(x, y) = \nabla_y G(x, y) \)

\[ \begin{aligned}
&= -y \cdot \frac{1}{n \sigma(n) |y-x|^n} \left[ (1|x-1|)y - |x| x + x \right] \\
&= -\frac{1}{n \sigma(n) |y-x|^n} \left[ 1 - |x| + |x| \frac{x}{|y|} y - xy \right] \\
&= \frac{1}{n \sigma(n) |y-x|^n} \left[ 1 - |x| + \frac{x}{|y|} y - xy \right] = \frac{1}{n \sigma(n) |y-x|^n} (1-|x|)(\frac{x}{|y|} + 1) \\
\Rightarrow u(x) = -\int_{\partial B(0,1)} g(y) \frac{1}{n \sigma(n) |y-x|^n} (1-|x|)(\frac{x}{|y|} + 1) \, dS(y) \\
&= -\frac{1}{n \sigma(n)} \int_{\partial B(0,1)} g(y) \frac{-|x| + |x| \frac{x}{|y|} y - xy}{|y| - x} \, dS(y) \\
&= -\frac{1}{n \sigma(n)} \int_{\partial B(0,1)} g(y) \frac{1}{|y-x|^n} (1-|x|)(\frac{x}{|y|} + 1) \, dS(y) \\
\end{aligned} \]