H-M exercises 6.1

Exercise 6.1. Complete the proof of Proposition 6.4. For (A), start by analyzing the characteristic polynomial of the linearized system. For (B), be sure to check points where the vector field is tangent to $E^+$ or $E^-$. Recall (6.3).

Proposition 6.4.

(A) For each $\epsilon > 0$, the system (6.2) has a one-dimensional unstable manifold $U_{c,\epsilon}$ and a two-dimensional stable manifold $S_{c,\epsilon}$ at $(0,0,0)$. $U_{c,\epsilon}$ has a component $U_{c,\epsilon}^+$ which is initially in the positive octant. If $(u,v,w)$ is a solution on $U_{c,\epsilon}^+$ then $u$, $v$, and $w$ are positive on every interval $(-\infty,T]$ in which $u \leq a$. Further, $w > 0$ in every interval $(-\infty,\tau]$ in which $u \geq 0$.

(B) The regions

$$E^+ = \{(u,v,w)|u < 1, v > 0, w > f(u)\}$$

$$E^- = \{(u,v,w)|u < 0, v < 0, w < f(u)\}$$

are each positively invariant for (6.2). Solutions $(u(s),v(s),w(s))$ entering $E^-$ tend to $-\infty$ in $u$ and $v$ as $s$ increases, while solution enterering $E^+$ tend to $+\infty$ in both variables. Conversely, a solution such that $u(s) \to -\infty$ eventually enters $E^-$, while a solution such that $u(s) \to \infty$ enters $E^+$.

Hints (A): The first sentence in the proof on p. 80 refers to Exercise 6.3, which must be a typo: Exercise 6.1 is intended. what is expected in part A of this exercise is a “routine stability analysis”.

The proposition concerns the system (6.2):

$$u' = v$$
$$v' = cv - f(u) + w$$
$$w' = \epsilon(c(u - \gamma w))$$

where $f(u) = u(u-a)(1-u)$ so $f'(0) = -a$. 

1
Let \( J(u, v, w) \) denote the Jacobian of the right side.

Calculate \( J_0 = J(0, 0, 0) \) and write out the characteristic polynomial \( p(\lambda) = \det(\lambda I - J_0) \).

Verify that \( p(0) < 0 \) and that \( p(\lambda) > 0 \) for \( \lambda \) large positive.

Infer that there is at least one positive eigenvalue \( \lambda = \lambda_1 > 0 \).

Let \( \lambda_2, \lambda_3 \) denote the other roots of \( p(\lambda) = 0 \). Then

\[
p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)
\]

\[
= \lambda^3 + \lambda^2(-\lambda_1 - \lambda_2 - \lambda_3) + \lambda(\lambda_2\lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3) - \lambda_1\lambda_2\lambda_3
\]

Write out the algebraic equations for \( \lambda_1, \lambda_2, \lambda_3 \) obtained by equating the powers of \( \lambda \) in this expression for \( p(\lambda) \) and the expression for \( p(\lambda) \) obtained from \( p(\lambda) = \det(\lambda I - J_0) \).

From one of these equations and from \( \lambda_1 > 0 \), infer that \( \lambda_2\lambda_3 > 0 \), so that if real, \( \lambda_2 \) and \( \lambda_3 \) must have the same sign. From another of these equations show that if \( \lambda_2 \) and \( \lambda_3 \) are real and of the same sign, there is a contradiction if they are both positive.

Then consider the complex case \( \lambda_2 = \mu + i\omega, \lambda_3 = \mu - i\omega \).

Hints (B)

The boundaries of \( E^+ \) are points where one or more of the inequalities in the definition \( E^+ = \{(u, v, w) | u < 1, v > 0, w > f(u)\} \) are replaced with equalities.

In examining the boundaries of \( E^+ \), the partial proof of proposition 6.4 in the text considers the point \((u, v, w) = (1, 0, 0)\) in which all three inequalities are replaced with equalities. The cases not treated are then (two equalities)

\[
u = 1, v = 0, w > f(u),
\]

\[
u = 1, v > 0, w = f(u),
\]

\[
u < 1, v = 0, w = f(u),
\]

and (one equality)

\[
u = 1, v > 0, w > f(u),
\]

\[
u < 1, v = 0, w > f(u),
\]

\[
u < 1, v > 0, w = f(u).
\]
The boundaries of \( E^- \) are points where one or more of the inequalities in the definition \( E^- = \{(u, v, w)|u < 0, v < 0, w < f(u)\} \) are replaced with equalities. The cases are then (three equalities)

\[ u = 0, v = 0, w = f(u), \]

(two equalities)

\[ u = 0, v = 0, w < f(u), \]
\[ u = 0, v < 0, w < f(u), \]
\[ u < 0, v = 0, w = f(u), \]

and (one equality)

\[ u = 0, v < 0, w < f(u), \]
\[ u < 0, v = 0, w < f(u), \]
\[ u < 0, v < 0, w = f(u), \]

In the partial proof of proposition 6.4 in the text, only this last case \( u < 0, v < 0, w = f(u) \) is considered.