H-M exercises 5.1

Exercise 5.1 Complete the proof of Lemma 5.1 by proving (iii) more generally.

Lemma 5.1. Let $a$ and $b$ be continuously differentiable functions in the closed interval $J = [0, 1]$, with $a(x) > 0$ and $b(x) \geq 0$. Let $u''$ be continuous on $J$ and suppose that in the open interval $J = (0, 1)$,

$$L(u) \equiv \frac{d}{dx} \left( a(x) \frac{du}{dx} \right) - b(x)u \leq 0.$$  

Then

(i) the function $u$ does not attain a negative minimum in the open interval $J$ unless $b$ is identically zero and $u$ is constant;

(ii) if $u \geq 0$ on $J$ and $u$ is not identically zero on $J$, then $u > 0$ on $J$;

(iii) if $u$ is not identically zero and $\inf_J u = 0$, then $u = 0$ at either 0 or 1, and if $u(0) = 0$ then $u'(0) > 0$ while if $u(1) = 0$ then $u'(1) < 0$.

The text gives the proofs of (i) and (ii), and also proves a weaker version of (iii) which may be rewritten (in terms of a function $w$) as follows:

Lemma 5.1$. Let $a$ and $b$ be continuously differentiable functions in the closed interval $J = [0, 1]$, with $a(x) > 0$ and $b(x) \geq 0$. Let $w''$ be continuous on $J$ and suppose that in the open interval $J = (0, 1)$,

$$\frac{d}{dx} \left( a(x) \frac{dw}{dx} \right) - b(x)w < 0.$$  

Then

(iii$_w$) if $w$ is not identically zero and $\inf_J w = 0$, then $w = 0$ at either 0 or 1, and if $w(0) = 0$ then $w'(0) > 0$ while if $w(1) = 0$ then $w'(1) < 0$.

Exercise 5.1 asks you to prove (iii) under the assumption

$$\frac{d}{dx} \left( a(x) \frac{du}{dx} \right) - b(x)u \leq 0$$

and in your proof you may use (i), (ii) and (iii$_w$).
Suppose $u$ satisfies the conditions of (iii). Then $u$ is not identically zero and \( \inf_J u = 0 \), so we know \( u(\eta) > 0 \) at some \( 0 < \eta < 1 \).

Since \( 0 = \inf_J u \) is the greatest lower bound for the values of $u$, for each number \( k = 1, 2, \ldots \), \( 1/k > 0 \) and so $1/k$ is not a lower bound: there are points \( \{z_k\}, k = 1, 2, \ldots \) such that \( u(z_k) < \frac{1}{k} \).

Show that there is a convergent subsequence \( z_{ki}, i = 1, 2, \ldots \) Let \( z_0 \) denote the limit of the convergent subsequence, and show \( u(z_0) = 0 \).

Use (ii) to show that \( 0 < z_0 < 1 \) leads to a contradiction, so that either \( z_0 = 0 \) or \( z_0 = 1 \).

Suppose \( z_0 = 0 \), so \( u(0) = 0 \). We wish to show \( u'(0) > 0 \).

Work with \( w(x) = u(x) + \epsilon v(x) \) for some positive $\epsilon$ to be determined, and for some function $v(x)$ to be determined.

If it can be shown that \( w \) satisfies (iii) on some interval, then the argument \( w'(0) > 0 \) implies \( u'(0) = w'(0) - \epsilon v'(0) > 0 \) will “work” provided \( v'(0) < 0 \).

If as on p.64 we use \( v(x) = 1 - e^{\kappa(x-\zeta)} \) with \( \kappa > 0 \), then \( v'(0) = -\kappa e^{-\zeta} < 0 \)

On p. 64 the point \( \zeta \) was chosen as a point where $u$ attained the minimum value \( u(\zeta) = 0 \). Here to analyze the left endpoint choose instead \( \zeta = 0 \).

\( \kappa \) can be chosen large enough so that

\[
L(v) = \frac{d}{dx} \left( a(x) \frac{dv}{dx} \right) - b(x)v < 0
\]

on \( J \) (show this)

Let \( w(x) = u(x) + \epsilon v(x) \), where $\epsilon > 0$ is small enough so that \( w(\eta) = u(\eta) + \epsilon v(\eta) > 0 \).

Now apply the parts of Lemma 5.1 that are already proven, to \( w \).