Exercise 2.13
Show that if $\beta > 0$ then the boundary value problem

$$f''' + f f'' + \beta (1 - f^2) = 0,$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

has a solution such that $f' > 0$ on $(0, \infty)$.

Hints: Work with $\beta = 1$, considering

$$f''' + f f'' + (1 - f^2) = 0,$$

with $f(0) = f'(0) = 0$ and $f''(0) = \alpha$, and leave the case of general positive $\beta$ for later, once the special case has been analyzed.

The numerical solutions provide guidance in choosing the sets $A$ and $B$.

From the numerical solution for $\alpha = 1.2$, the graph of $f'$ has a maximum value at some time $t > 0$ and then decreases. Also, the maximum value obeys $f' < 1$, so the graph of $f'$ does not cross over the line $f' = 1$.

From the numerical solution for $\alpha = 1.25$, the graph of $f'$ crosses the line $f' = 1$, and once $f' > 1$, $f'$ continues to increase.

Answer the following questions:

a) if $f'$ attains a maximum value at some point $t_0$ with $f'(t_0) < 1$, can $f'$ cross into $f' > 1$ at some later time?

b) if $f'$ has entered the region $f' > 1$ so $f(t_1) > 1$ at some point $t_1$, can $f'$ cross back into $f' < 1$ at some later time?

Roughly, $A$ will be the set of $\alpha > 0$ with $\alpha$ small, such that the solution $f'$ has a maximum in the region $f' < 1$ and subsequently decreases. The actual definition of $A$ should be in terms of condition(s) on $f'$ and/or $f''$ involving only strict inequalities. Let “property P” denote this set of conditions on the solutions.

Roughly, $B$ will be the set of $\alpha > 0$ with $\alpha$ large, such that the solution $f'$ crosses into the region $f' > 1$ and subsequently increases. The actual definition of $B$ should be in terms of conditions on $f'$ and/or $f''$ involving only strict inequalities. Let “property Q” denote this set of conditions on the solutions.
If A and B chosen correctly, it will be possible to show that A and B are each open by using continuity of the solution of the initial value problem with respect to $\alpha$.

That A and B are disjoint will follow from the answers to the questions a) and b) above.

Show A is nonempty by calculating the first few terms of the Taylor expansion of the solution about $t = 0$. For $\alpha = 0$, $f'$ immediately becomes negative but for $\alpha$ positive $f'$ starts out positive. Use these properties to show the existence of positive $\alpha$ such that $f'$ attains a maximum value at some small positive $t$.

Show B is nonempty by showing that for sufficiently large $\alpha$, $f'$ crosses into $f' > 1$. Let $F = \int_0^t f(s)ds$. Then $(e^F f''')' = e^F (f''' + ff'') = e^F((f')^2 - 1)$ and

$$f''(t) = e^{-F(t)}[\alpha + \int_{s=0}^t e^{F(s)}((f'(s))^2 - 1)ds]$$

which may help in showing that $f''$ remains large long enough so that $f'(t) = \int_{s=0}^t f''(s)ds$ enters $f' > 1$.

Argue that since A and B are disjoint and open, $A \cup B \neq R^+$ so there is an $\alpha = \alpha^* \in R^+$ that belongs to neither A nor B.

The solution for $\alpha = \alpha^*$ has neither property P nor property Q: that is, $f'$ is increasing but does not cross into $f' > 1$.

Let $\omega$ denote the right end endpoint of the forward interval of existence $[0, \omega)$ for this solution. Show that the assumption $\omega$ finite, leads to a contradiction.

Let $L = \lim_{t \to \infty} f'(t)$. Then $L \leq 1$. Show that the assumption $L < 1$ leads to a contradiction.

Results from Exercise 2.5 may be useful:

2.5b) If $q''$ is continuous on $[0, \infty)$ and $\lim_{s \to \infty} q(s)$ exists, but $\lim_{s \to \infty} q'(s)$ does not exist, then $q''$ is unbounded.

2.5d) If $\lim_{s \to \infty} q''(s) = 0$ and $q$ is bounded, then $\lim_{s \to \infty} q'(s) = 0$. 

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