NAME (please print legibly): ____________________________________________
Your University ID Number: __________________________________________

- No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

The Black-Scholes formulas are

\[ C_{\text{euro}}(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P_{\text{euro}}(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

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1. (10 points) Consider a European-style option \( V(S,t) \) with payoff at expiry given by

\[
\Lambda(S(T)) = S(T).
\]

Let \( V_0 \) be the time-zero value of this option. Give a complete no-arbitrage argument to explain why \( V_0 \) must equal \( S_0 \). In each case, describe all actions taken at \( t = 0 \) and at \( t = T \), and explain how these actions lead to an arbitrage. Finally, after considering both cases, explain how you draw the conclusion.

a) Consider the case \( V_0 < S_0 \):

At \( t=0 \), short asset at \( S_0 \), and buy option at \( V_0 \).

At \( t=T \), exercise option to get payoff \( S(T) \), and buy back the asset at \( S(T) \).

Profit:

\[
+ S_0 - V_0 + S(T) - S(T) = S_0 - V_0 > 0.
\]

Arbitrage.

b) Consider the case \( V_0 > S_0 \):

At \( t=0 \), sell option at \( V_0 \), and buy asset at \( S_0 \).

At \( t=T \), sell asset at \( S(T) \), and close the option at \( S(T) \).

Profit:

\[
V_0 - S_0 + S(T) - S(T) = V_0 - S_0 > 0.
\]

Arbitrage.

c) After considering both cases, we see that:

From (a), we have \( V_0 > S_0 \).

From (b), we have \( V_0 < S_0 \).

So, we have \( V_0 = S_0 \).
2. (10 points) (559 only) The value of an asset $S$ at time $T$ is modeled as a random variable $x$ with probability density $f(x) = 0$ for $x < 0$, $f(x) = xe^{-x^2/2}$ for $0 \leq x < \infty$.

a) Calculate the expected value $E[S(T)]$. Hint: $d(-e^{-x^2/2}) = xe^{-x^2/2}dx$, $\int u\,dv = uv - \int v\,du$.

$$\Lambda^C(x) = \begin{cases} x - K, & x \geq K \\ 0, & x < K \end{cases}$$

$$\Lambda^P(x) = \begin{cases} 0, & x \geq K \\ K - x, & x < K \end{cases}$$

b) Find the expected payoff $E[\Lambda^C(S(T))]$. You may assume that $N(z) + N(-z) = 1$.

c) Show $\Lambda^C(x) - \Lambda^P(x) = x - K$, and use this identity with the results from a) and b) to find $E[\Lambda^P(S(T))]$.

\[ \begin{align*}
\mathbb{E}[S(T)] &= \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} x \cdot 0 \, dx + \int_{0}^{\infty} xe^{-x^2/2} \, dx \\
&= -xe^{-x^2/2} \Big|_0^\infty + \int_{0}^{\infty} e^{-x^2/2} \, dx \\
&= \frac{-e^{-x^2/2}}{x} \Big|_0^\infty + \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2} \, dx \\
&= \lim_{x \to \infty} \left( \frac{x}{\sqrt{2\pi}} \right) + 0 + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \\
&= \frac{\sqrt{2\pi}}{2} = \frac{\sqrt{\pi}}{2}.
\end{align*} \]

\[ \mathbb{E}[\Lambda^C(S(T))] = \int_{\infty}^{\infty} \Lambda^C(x) f(x) \, dx = \int_{0}^{\infty} \Lambda^C(x) \pi e^{-x^2/2} \, dx \\
= \int_{K}^{\infty} (x-K) \cdot xe^{-x^2/2} \, dx = \int_{K}^{\infty} (x-K) \cdot d(-e^{-x^2/2}) \\
= -e^{-x^2/2} \Big|_{K}^{\infty} + \int_{K}^{\infty} e^{-x^2/2} \, dx - (x-K) e^{-x^2/2} \Big|_{K}^{\infty} = 0 + \int_{K}^{\infty} e^{-x^2/2} \, dx \\
= \sqrt{\pi} \cdot \int_{K}^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi} \cdot N(-K) = \sqrt{\pi} \left( 1 - N(K) \right).
\]

\[ \text{c) See back side.} \]
If \( x > K \), \( \Lambda^c(x) = x - K \) and \( \Lambda^p(x) = 0 \).

\[
\Rightarrow \Lambda^c(x) - \Lambda^p(x) = x - K
\]

If \( x < K \), \( \Lambda^c(x) = 0 \), \( \Lambda^p(x) = K - x \).

\[
\Rightarrow \Lambda^c(x) - \Lambda^p(x) = 0 - (K - x) = x - K.
\]

So, for both cases, \( \Lambda^c(x) - \Lambda^p(x) = x - K \).

\[
E[\Lambda^c(x) - \Lambda^p(x)] = E[x-K]
\]

\[
\Rightarrow E[\Lambda^p(x)] = E[\Lambda^c(x)] - E[x] + K
\]

\[
= E[\Lambda^c(S(n))] - E[S(n)] + K
\]

\[
= \sqrt{2}\pi \left(1 - N(K)\right) - \frac{\sqrt{2}\pi}{2} + K
\]

\[
= \frac{\sqrt{2}\pi}{2} - \sqrt{2}\pi N(K) + K.
\]
3. (10 points) For an asset with value $S$ at time $t$, the Black-Scholes value for an option with payoff $\Lambda(S(T))$ is

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{y=0}^{\infty} \Lambda(y) e^{-\frac{(\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}} \frac{dy}{y}$$

a) Consider the change of integration variable from $y$ to $x = \frac{\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$. Calculate the differential $dx$ in terms of $dy$. Analyze the limits $y \to 0^+$ and $y \to \infty$ of the change of variable. Show that

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \Lambda(e^{x\sigma \sqrt{T-t} + \ln S + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{x^2}{2}} dx$$

b) Use the result from a) to represent the Black-Scholes value for an asset-or-nothing put with payoff

$$\Lambda^P(y) = \begin{cases} 0, & y > K \\ y, & y \leq K \end{cases}$$

as an integral. Express the limit(s) of integration in terms of quantities defined on p. 1. However, you DO NOT need to work the integral.

\[ d\chi = d\left(\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)\right) = \frac{dy}{y} \]

\[ x = \ln y - \ln S + \frac{\sigma^2}{2} (T-t) \]

\[ \Rightarrow y = e^{x\sigma \sqrt{T-t} + \ln S + (r - \frac{\sigma^2}{2})(T-t)} \]

\[ y \to 0^+, \Rightarrow \ln y \to -\infty, \quad \text{then} \quad x \to -\infty. \]

\[ y \to \infty, \Rightarrow \ln y \to \infty, \quad \text{then} \quad x \to +\infty. \]

\[ W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{x=-\infty}^{\infty} \Lambda(e^{x\sigma \sqrt{T-t} + \ln S + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{x^2}{2}} dx \]

\[ = e^{-r(T-t)} \int_{x=-\infty}^{\ln K - \ln S - \frac{\sigma^2}{2} (T-t)} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \]

\[ W(S, t) = e^{-r(T-t)} \int_{\ln K - \ln S - \frac{\sigma^2}{2} (T-t)}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \]

\[ = e^{-r(T-t)} \int_{-\infty}^{\ln K - \ln S - \frac{\sigma^2}{2} (T-t)} e^{-\frac{1}{2}(x - \sigma \sqrt{T-t})^2 + \frac{1}{2} \sigma^2 (T-t) + \ln S + (r - \frac{\sigma^2}{2})(T-t)} dx \]
4. (10 points) a) Assume identities \( N'(z) + N'(-z) = 1 \) and \( SN'(d_1) - e^{-(T-t)}KN'(d_2) = 0 \) are known. Use them to show i) \( N'(-z) = N'(z) \), and ii) \( e^{-(T-t)}KN'(-d_2) - SN'(-d_1) = 0 \).

b) Show that for \( P_{\text{euro}}(S, t) \) from page 1,
\[
\frac{\partial P_{\text{euro}}}{\partial S} = N(-d_1).
\]
Hint: use ii) and \( \frac{\partial A}{\partial S} = \frac{\partial A}{\partial S} \).

c) A portfolio \( \Pi \) consists of a unit of the asset \( S \) and a quantity \( b_0 \) of put options \( P_{\text{euro}}(S, t) \) such that \( \frac{\partial \Pi}{\partial S} \), the delta of the portfolio, is zero at \( t = 0 \). Find \( b_0 \).

d) As \( t \to T^- \), the put option is in the money. As \( t \to T^- \), what is the limit value of \( d_1 \), and what is the limit of the delta of the portfolio?

\[
\begin{align*}
\text{(a)} & \quad N(z) + N(-z) = 1 \quad \Rightarrow \quad \frac{d}{dz} \left( N(z) + N(-z) \right) = 0 \quad \Rightarrow \quad N'(z) + N'(-z) \cdot \frac{d(-z)}{dz} = 0 \\
& \quad \Rightarrow \quad N'(z) - N'(-z) = 0 \\
& \quad \Rightarrow \quad N'(z) = N'(-z).
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad e^{-r(T-t)}KN'(-d_1) - SN'(-d_1) \\
& \quad = e^{-r(T-t)}KN'(d_2) - SN'(d_1) \\
& \quad = 0. \quad \text{Since} \quad SN'(d_1) - e^{-r(T-t)}KN'(d_2) = 0.
\end{align*}
\]

\[
\begin{align*}
\text{(c)} & \quad \frac{\partial P_{\text{euro}}}{\partial S} = KE^{-r(T-t)}N'(d_1) \frac{\partial d_2}{\partial S} - N(-d_1) - S \cdot \frac{\partial N(-d_1)}{\partial d_1} \cdot \frac{d(-d_1)}{dS} \\
& \quad = KE^{-r(T-t)}N'(d_2) \frac{\partial d_2}{\partial S} - (N(-d_1) + S \cdot N'(-d_1) \frac{\partial d_1}{\partial S}) \\
& \quad = \frac{\partial d_1}{\partial S} \left( -KE^{-r(T-t)}N'(d_2) + SN'(d_1) \right) = N(-d_1) \\
& \quad = -N(-d_1).
\end{align*}
\]

\[
\begin{align*}
\text{(d)} & \quad \Pi = S + b_0 \cdot P_{\text{euro}}(S, t). \\
& \quad \frac{\partial \Pi}{\partial S} = 1 + b_0 \frac{\partial P_{\text{euro}}}{\partial S} = 1 - b_0 N(-d_1) \quad \text{where} \quad d_1 \text{ at } t = 0.
\end{align*}
\]
As $t \to T^-$,

$$\lim_{t \to T^-} d_1 = \lim_{t \to T^-} \frac{\ln S + \left( r + \frac{1}{2} \sigma^2 \right) (T-t) - \ln K}{\sigma \sqrt{T-t}}$$

as $t \to T^-$, $\frac{1}{\sigma \sqrt{T-t}} \to +\infty$.

Put option is in the money, so $S < K$.

$\Rightarrow \frac{\ln S}{K} < \ln 1 = 0$

and $(r + \frac{1}{2} \sigma^2) (T-t) \to 0^+$ as $t \to T^-$. 

$\Rightarrow \lim_{t \to T^-} d_1 = -\infty$.

$$\lim_{t \to T^-} \frac{\partial V}{\partial S} = \lim_{t \to T^-} \left( 1 - b_0 N(-d_1) \right)$$

Since $d_1 \to -\infty$, $-d_1 \to +\infty \Rightarrow N(-d_1) \to 1$

$\Rightarrow \lim_{t \to T^-} \frac{\partial V}{\partial S} = 1 - b_0 \cdot 1 = 1 - b_0$. 

(\text{with})