MATH 559
Final Exam
May 13, 2016

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

- Please try all questions. Questions are equally weighted. No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring in a formula sheet, written both sides with whatever they please. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Sharing of calculators is not allowed. Please show all work. Correct answers without supporting work may not receive full credit. You may use back pages if necessary.

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
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1. **(10 points)** (559 only) The stock Xilinx had values as in the table below.

<table>
<thead>
<tr>
<th>Date</th>
<th>i</th>
<th>$S_i$</th>
<th>$S_i / S_{i-1}$</th>
<th>$U_i = \ln(S_i / S_{i-1})$</th>
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<tr>
<td>04/27/16</td>
<td>6</td>
<td>47.35</td>
<td>1.019</td>
<td>0.02</td>
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<td>04/20/16</td>
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<td>46.46</td>
<td>0.996</td>
<td>0.00</td>
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<td>0.987</td>
<td>-0.01</td>
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<td>47.25</td>
<td>0.987</td>
<td>-0.01</td>
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<td>1.011</td>
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<td>0.993</td>
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<tr>
<td>03/16/16</td>
<td>0</td>
<td>47.65</td>
<td>..</td>
<td>..</td>
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</table>

Calculate the mean $a_M$ and standard deviation $b_M$ of the data $U_i = \ln(S_i / S_{i-1}), \ i = 1..M$ and use the results to approximate the volatility of the asset $S_i$.

Work starting with the given values $U_i$: do NOT recompute these values.
2. (10 points) (559 only) a) Let \( r_F = \frac{1}{10} z_1 \) and \( r_G = \frac{4}{100} z_1 + \frac{3}{100} z_2 \), where \( z_1 \) and \( z_2 \) are independent random variables each with mean 0 and standard deviation 1.

Let \( \sigma_F \equiv \sqrt{\text{var}(r_F)} \), \( \sigma_G \equiv \sqrt{\text{var}(r_G)} \). Show that \( \sigma_F = \frac{1}{10} \), \( \sigma_G = \frac{5}{100} \) and \( \text{cov}(r_G, r_F) = \frac{4}{1000} \).

b) We model the prices at \( T = \frac{1}{2} \) year from now, of orange and of grapefruit juice, as \( F(T) = F_0 (1 + \frac{r_F}{2}) \) and \( G(T) = G_0 (1 + \frac{r_G}{2}) \), respectively.

What are \( \text{var}(F(T)) \), \( \text{var}(G(T)) \), and \( \text{cov}(G(T), F(T)) \) ?

c) Farmer J will have a crop of grapefruit that will be ready for harvest and sale as 100,000 pounds of grapefruit juice in \( \frac{1}{2} \) year. J is worried about possible price changes and so considers minimum variance hedging. There is no futures contract for grapefruit juice, but there is a future contract for orange juice. The current prices are \( F_0 = 1.20 \) per pound for orange juice and \( G_0 = 1.40 \) per pound for grapefruit juice.

J considers the hedged position \( y(h) = G(T) - h(F(T) - F_0) \).

For what value \( h = h_{\text{min}} \) is the minimum variance of \( y(h) \) attained?

d) Show that \( y(h_{\text{min}}) = c_0 + c_1 z_1 + c_2 z_2 \) for some constants \( c_0, c_1, c_2 \) and find these constants.
3. **(10 points)** a) Let \( V(S,t;K) \) be the value of an option with strike \( K \), that satisfies the Black-Scholes equation

\[
\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - rV = 0
\]

Show that \( f = \frac{\partial V}{\partial K} \) also satisfies the Black-Scholes equation.

b) Let \( f(S,t;K) = \frac{\partial}{\partial K} P_{\text{euro}}(S,t;K) \). Calculate \( f(S(T),T;K) = \frac{\partial}{\partial K} \max(K - S(T),0) \) by considering the two cases \( S(T) < K, S(T) > K \). **Do not** use the B-S formulas.

c) Let \( P_{\text{cash}}(S,t;K) \) be the Euro-style option that pays 1 unit of cash if \( S(T) < K \) and pays nothing if \( S(T) > K \). From a) and b) above, we infer that \( P_{\text{cash}} = f = \frac{\partial}{\partial K} P_{\text{euro}} \). Find the formula for \( P_{\text{cash}}(S,t;K) \) by differentiating the Black-Scholes formula

\[
P_{\text{euro}}(S,t;K) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \quad \text{where}
\]

\[
d_1 = \frac{\ln S + (r + \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma\sqrt{T-t}}
\]

\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx
\]

Assume without proof that \( Ke^{-r(T-t)}N'(d_2) - SN'(d_1) = 0 \).
4. (10 points) (559 only) The Black-Scholes formulas are

\[ C^\text{euro}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P^\text{euro}(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2} \sigma^2)(T - t) - \ln K}{\sigma \sqrt{T - t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2} \sigma^2)(T - t) - \ln K}{\sigma \sqrt{T - t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

Assume without proof that \( SN'(-d_1) - Ke^{-r(T-t)}N'(-d_2) = 0 \).

a) Calculate \( \lim_{\sigma \to 0^+} C^\text{euro}(S,t;\sigma) \)

b) Calculate \( \lim_{\sigma \to \infty} C^\text{euro}(S,t;\sigma) \)

c) Show \( \frac{\partial}{\partial \sigma} C^\text{euro}(S,t;\sigma) < 0 \).

(It follows that if one knows an actual option value \( C^* \) falling between the limiting values in a) and b), the equation \( C^\text{euro}(S,t;\sigma) = C^* \) can be solved to determine volatility)
5. (10 points) At time $t_0 < T$, where $T$ is expiration, you hold an American style call option with strike $K$.

a) Suppose $S(t_0) > K$. If you exercise the option at $t_0$, what is your profit or loss?

b) Suppose $S(t_0) > K$. You short one unit of the asset, receiving $S(t_0)$ in cash. Of this cash, you pocket $S(t_0) - Ke^{-r(T-t_0)}$ and deposit $Ke^{-r(T-t_0)}$ in a bank at the risk-free rate $r$. At time $T$, you withdraw $K$ from the bank.
If $S(T) > K$, you exercise the call option to purchase the asset for $K$.
If $S(T) \leq K$, you purchase the asset on the spot market for $S(T)$.
You return the asset to the owner.
Using this strategy, what is your profit or loss? Express the result in terms of cash at $t_0$, using the discount factor $e^{-r(T-t_0)}$ to convert cash at $t = T$ to cash at $t_0$.

c) Show that the scheme described in b) is more profitable than the scheme in a).

d) Since an American call is not exercised early, $C^{\text{amer}}(S,t;K) = C^{\text{euro}}(S,t;K)$ and so

$$C^{\text{amer}} + Ke^{-r(T-t)} = C^{\text{euro}} + Ke^{-r(T-t)} = P^{\text{euro}} + S$$

by the put-call parity formula for Euro options. Which is worth more, an American put or a Euro put with the same strike and expiration? Circle the correct inequality below:

$$C^{\text{amer}} + Ke^{-r(T-t)} \geq P^{\text{amer}} + S, \quad C^{\text{amer}} + Ke^{-r(T-t)} \leq P^{\text{amer}} + S$$
6. (10 points) (559 only) The figure shows a portion of the asset binomial tree used in calculation of $P(S_0, 0)$ for an American put option $P(S, t)$ with expiration $T = 1$. The asset pays a dividend. The different symbols show the put option “exercise” and “do not exercise” points found in the calculation.

a) For (roughly) what initial asset $S_0$ and strike price $K$ was the tree constructed, and at what time $t_d$ is the dividend paid?

b) Write in the values of the put option along the top ($t = T = 1$) boundary and indicate the point $S = K$. Write in the values of the put option along the left ($S = 0$) boundary.

c) Label on the figure, the region where the option obeys $P(S, t) = K - S$ as “exercise”. Label the region where $P(S, t) > K - S$ as “do not exercise”. Extend the boundaries between “exercise” and “do not exercise” into the whitespace.

d) Sketch on the figure, the optimal exercise boundary $S = S^*(t), 0 \leq t \leq T$. 
7. **(10 points)** Match the option payoffs defined below with the letters labeling their equivalent option names.

1. \( \max(S(T) - K, 0) \)  
   A  Lookback Call

2. \( \max(K - S(T), 0) \)  
   B  Asian Put

3. \( \max_{0<t<T} (S(t) - K, 0) \)  
   C  Amer Call

4. \( \max_{0<t<T} (K - S(t), 0) \)  
   D  Bermudan Call

5. \( \max_{t=T/2, t=T} (S(t) - K, 0) \)  
   E  Euro Call

6. \( \max_{t=T/2, t=T} (K - S(t), 0) \)  
   F  Amer Put

7. \( \max(S_{max} - K, 0) \) where \( S_{max} = \max_{0<t<T} S(t) \)  
   G  Bermudan Put

8. \( \max(K - S_{avg}, 0) \) where \( S_{avg} = \frac{1}{T} \int_0^T S(t) dt \)  
   H  Euro Put
8. (10 points)

Suppose \( f(x) \) is given, and \( X \) is a random variable from the uniform distribution on \([-1, 1]\).

a) In the basic Monte-Carlo computation to approximate \( E[f(X)] \), values \( X_i \) are sampled and the approximation used is \( E[f(X)] \approx \frac{1}{M} \sum_{i=1}^{M} f(X_i) \)

Give a 95% confidence interval for this approximation.

Write the standard deviation as \( \sigma_{MC} = \sqrt{\text{var}(f(X))} \).

b) In the antithetic Monte-Carlo computation to approximate to \( E[f(X)] \) with antithetic variable \(-X\), what is the approximation to \( E[f(X)] \)?

Give a 95% confidence interval for this approximation.

Write the standard deviation as \( \sigma_{anti} = \sqrt{\text{var}(\frac{f(X)+f(-X)}{2})} \).

c) Assume \( \text{var}(\frac{f(X)+f(-X)}{2}) = \frac{1}{4} [\text{var}(f(X)) + 2\text{cov}(f(X), f(-X)) + \text{var}(f(-X))] \).

Show that if \( \text{cov}(f(X), f(-X)) < 0 \), then \( \text{var}(\frac{f(X)+f(-X)}{2}) < \frac{1}{2} \text{var}(f(X)) \).

d) For the function \( f(x) = e^x \), explain why \( \text{cov}(f(X), f(-X)) < 0 \).

e) For the function \( f(x) = e^x \) it turns out that \( \sigma_{MC}^2 = 0.432 \) and \( \sigma_{anti}^2 = 0.026 \).

What is the radius of the 95% confidence interval for 200 samples with basic Monte-Carlo?

What is the radius of the 95% confidence interval for 100 antithetic pairs?

What is the exact value of \( E[f(X)] \)?