NAME (please print legibly): ____________________________________________
Your University ID Number: ___________________________________________

- No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

The Black-Scholes formulas are

\[ C_{\text{euro}}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P_{\text{euro}}(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

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1. (10 points) An asset has value \( S_0 = 120 \) at \( t = 0 \), and will have unknown value \( S(T) \) at time \( T \). A call option with strike \( K = 100 \) is currently \( (t = 0) \) selling for \( C_0 = 21 \), and the discount factor for the period 0 to \( T \) is \( e^{-rT} = 0.99 = 99/100 \). The risk-free rate is \( r \).

a) Consider the portfolio obtained by selling (short) one unit of the asset at \( t = 0 \), buying one call option, and investing the present value of the strike \( K \) in a bank at the risk-free rate. Show there is an arbitrage opportunity. Give all actions at \( t = T \), and discuss the type of arbitrage.

\[
\begin{align*}
  t=0 & \quad \text{short one unit of the asset, we got} \quad 120, \\
  & \quad \text{buy one call option, we pay} 21, \text{then we} \quad \uparrow \\
  & \quad \text{invest} 99 \text{ in bank, at time} \ T, \text{we get} 99 \cdot \frac{100}{99} = 100 \text{ at } T \quad \Rightarrow \quad \text{No matter} S(T) > K \quad \text{or not at time} \ T, \text{we can at least use} \ 100 \text{ is cover our short.}
\end{align*}
\]

In details, if \( S(T) > 100 \), we are even, (call option, carry out) if \( S(T) < 100 \), we get 100 - \( S(T) \) (buy asset directly), so there is type B arbitrage, we have positive probability getting positive payout without any cost.

b) If \( C_0 = 20 \), we get invest 100 in bank, \( 1 \) at \( T \), we have at least \( 100 \cdot \frac{100}{99} = \frac{100}{99} \) payout without any cost from pocket, then we have type A arbitrage.

If \( C_0 = 22 \), then, we just get \( 100 \cdot \frac{100}{99} = \frac{100}{99} \) at time \( T \), less than the strike price. No arbitrage.
2. (10 points) (559 only) The value of an asset \( S \) at time \( T \) is modeled as a random variable \( x \) with probability density

\[
    f = \begin{cases} 
        0 & \text{for } x < 0, \\
        \frac{1}{5} & \text{for } 0 \leq x \leq 5, \\
        0 & \text{for } x > 5.
    \end{cases}
\]

Call and put options with strike \( K \) based on this asset have payoffs

\[
    \Lambda^C(x) = \begin{cases} 
        x - K, & x \geq K \\
        0, & x < K 
    \end{cases}, \quad \Lambda^P(x) = \begin{cases} 
        0, & x \geq K \\
        K - x, & x < K 
    \end{cases}
\]

a) Calculate \( E[S(T)] \), the expected value of \( S(T) \).

b) Find the expected payoff \( E[\Lambda^C(S(T))] \). Consider cases \( 0 \leq K \leq 5 \) and \( K > 5 \).

c) Show \( \Lambda^P(x) - \Lambda^C(x) = K - x \), and use this identity with the results from a) and b) to find \( E[\Lambda^P(S(T))] \).

\[
    \text{a) } E[S(T)] = \int_0^5 \frac{1}{5} x \cdot \frac{1}{5} dx = \frac{1}{10} \left( \frac{5^2}{2} + 0 \right) = \frac{5}{2} \\
    \text{b) When } 0 \leq K \leq 5 \\
    E[\Lambda^C(S(T))] = \int_K^\infty (x - K) \cdot \frac{1}{5} dx \\
    = \left. \frac{1}{5} \left( \frac{1}{2} x^2 - \frac{1}{2} K x \right) \right|_K^5 \\
    = \frac{1}{2} - K + \frac{1}{2} K^2 \\
    \text{c) } \Lambda^P(x) - \Lambda^C(x) = \begin{cases} 
        x - K, & x \geq K \\
        K - x, & x < K 
    \end{cases}
\]

\[
    \text{Then } E[\Lambda^P(S(T))] = E[\Lambda^C(S(T))] + K - E(S) \]

\[
    \text{When } 0 \leq K \leq 5, \quad E[\Lambda^P(S(T))] = \frac{1}{5} K^2 \\
    \text{When } K > 5, \quad E[\Lambda^P(S(T))] = K - \frac{5}{2} \]

\[
    \text{a) } 4/4 \quad \text{b) } 3/3 \quad \text{c) } 2/2
\]

\[
    \text{The expected payoff } E[\Lambda^P(S(T))] \text{ for call and put options.}
\]
3. (10 points) (559 only) For an asset with value $S$ at time $t$, the Black-Scholes value for an option with payoff $\Lambda(S(T))$ is:

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}\sigma^2(T-t)} \int_{y=0}^{\infty} \Lambda(y) e^{-\frac{(\ln y - \ln S - \frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} \frac{dy}{y}$$

a) Let $x = \frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$ and show that

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \Lambda(e^{x \sqrt{T-t} + \ln S + (r - \frac{1}{2}\sigma^2)(T-t)}) e^{-\frac{x^2}{2}} dx$$

b) Use the formula from a) to calculate the Black-Scholes value for an asset-or-nothing call with payoff

$$\Lambda(y) = \begin{cases} y, & y \geq K \\ 0, & y < K \end{cases}$$

Your answer should be written in terms of the function $N(z)$ given on p. 1.

a) $\alpha = \frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$, then

$$\frac{dy}{y} = \sigma \sqrt{T-t} \, d\alpha, \quad y = e^{\frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}}$$

Moreover, $y \to 0^+$, $\alpha \to -\infty$; $y \to \infty$, $\alpha \to \infty$.

Then $W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi} \sigma \sqrt{T-t}} \int_{\alpha=-\infty}^{\alpha=0} \Lambda(e^{\frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}}) e^{-\frac{\alpha^2}{2}} d\alpha$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi} \sigma \sqrt{T-t}} \int_{\alpha=-\infty}^{\alpha=0} \Lambda(e^{\frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}}) e^{-\frac{\alpha^2}{2}} d\alpha$$

b) $W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi} \sigma \sqrt{T-t}} \int_{y=k}^{y=\infty} \Lambda(y) e^{-\frac{(\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} \frac{dy}{y}$

Again, we let $\alpha = \frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$, then $y=k \Rightarrow \alpha = \frac{\ln S - \ln k - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi} \sigma \sqrt{T-t}} \int_{\alpha=-d}^{\alpha=-d} \Lambda(e^{\frac{\ln y - \ln S - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}}) e^{-\frac{\alpha^2}{2}} d\alpha$$

$$= S \frac{e^{-r(T-t)}}{\sqrt{2\pi} \sigma \sqrt{T-t}} \int_{\alpha=-d}^{\alpha=-d} e^{-\frac{\alpha^2}{2}} d\alpha.$$
\[ W(s,t) = \frac{1}{\sqrt{2\pi s}} \int_{-d_2}^{d_2} e^{-\frac{x^2}{2} + \sqrt{t-t^2} \cdot \alpha} \, dx \]

Note that: \[-\frac{\alpha^2}{2} + \sqrt{t-t^2} \cdot \alpha = -\frac{1}{2} (\alpha^2 - 2\sqrt{t-t^2} \cdot \alpha)\]

\[ = -\frac{1}{2} [(\alpha - \sqrt{t-t^2})^2 - \sqrt{t-t^2}] \]

\[ = -\frac{1}{2} (\alpha - \sqrt{t-t^2})^2 + \frac{\alpha^2}{2} (t-t^2) \]

So, \[ W(s,t) = \frac{1}{\sqrt{2\pi s}} \int_{-d_2}^{d_2} e^{-\frac{x^2}{2} (t-t^2)} \cdot e^{-\frac{(\alpha - \sqrt{t-t^2})^2}{2} (t-t^2)} \, dx \]

\[ = \frac{1}{\sqrt{2\pi s}} \int_{-d_2}^{d_2} e^{-\frac{(\alpha - \sqrt{t-t^2})^2}{2}} \, dx \]

Let \( \alpha - \sqrt{t-t^2} = z \), then \( dx = dz \).

When \( \alpha = d_2 \), \( z = d_2 - \alpha = -d_2 + \sqrt{t-t^2} = -d_1 \).

Then, \[ W(s,t) = \frac{1}{\sqrt{2\pi s}} \int_{-d_2}^{d_2} e^{-\frac{z^2}{2}} \, dz \]

\[ = \sqrt{\pi} S \left( 1 - \frac{1}{2} \int_{-d_1}^{d_1} e^{-\frac{z^2}{2}} \, dz \right) \]

\[ = \sqrt{\pi} S \cdot (1 - N(d_1)) \]

\[ = S \cdot N(d_1) \]
4. (10 points) a) Show that for the $d_1, d_2$ given on page 1, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$.

b) Assume the identity $SN'(d_1) e^{-r(T-t)} K N'(d_2)$ is known. Show that for $C^{\text{euro}}(S, t)$ given on page 1,

$$\frac{\partial C^{\text{euro}}}{\partial S} = N(d_1).$$

c) A portfolio II consists of a unit of the asset $S$ and a quantity $k_0$ of call options $C^{\text{euro}}(S, t)$ such that $\frac{\partial \Pi}{\partial S}$, the delta of the portfolio, is zero at $t = 0$. Find $k_0$.

d) As $t \to T^-$, the call option out of the money. What is the delta of the portfolio as $t \to T^-$?

a) $\frac{\partial d_1}{\partial S} = \frac{1}{\sqrt{17-t}} \cdot \frac{1}{5}$, $\frac{\partial d_2}{\partial S} = \frac{1}{\sqrt{17-t}} \cdot \frac{1}{5} = \frac{\partial d_1}{\partial S}$

b) $\frac{\partial C^{\text{euro}}}{\partial S} = N(d_1) + S \cdot N'(d_1) \frac{\partial d_1}{\partial S} - K e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}$

Since we have $SN'(d_1) - K e^{-r(T-t)} N'(d_2) = 0$, then

$$SN'(d_1) \frac{\partial d_1}{\partial S} - K e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S} = 0.$$

Thus, $\frac{\partial C^{\text{euro}}}{\partial S} = N(d_1)$

c) $\Pi = S + k_0 C^{\text{euro}}(S, t)$

$$\frac{\partial \Pi}{\partial S} = 1 + k_0 N(d_1), \text{ when } t = 0, \frac{\partial \Pi}{\partial S} = 0, \text{ thus we have } 1 + k_0 N(d_1(t=0)) = 0, \text{ then } k_0 = -\frac{1}{N(d_1(t=0))}$

d) Recall $d_1 = \frac{\ln S - \ln K + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$

$t \to T^-$, call option out of the money, then $S < K$, $\ln S - \ln K < 0$.

Then, as $t \to T^-$, $d_1 \to -\infty$, and $N(d_1) \to 0$

Thus, $\Delta = \frac{\partial \Pi}{\partial S} = 1 + k_0 \cdot 0 = 1$