NAME (please print legibly): ____________________________________________
Your University ID Number: ____________________________________________

- No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

The Black-Scholes formulas are

\[ C^{\text{euro}}(S,t) = S N(d_1) - K e^{-r(T-t)} N(d_2), \quad P^{\text{euro}}(S,t) = K e^{-r(T-t)} N(-d_2) - S N(-d_1), \]

where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

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1. **(10 points)** An asset has value $S_0 = $120 at $t = 0$, and will have unknown value $S(T)$ at time $T$. A call option with strike $K = $100 is currently ($t = 0$) selling for $C_0 = $21, and the discount factor for the period 0 to $T$ is $e^{-rT} = 0.99 = 99/100$. The risk-free rate is $r$.

a) Consider the portfolio obtained by selling (short) one unit of the asset at $t = 0$, buying one call option, and investing the present value of the strike $K$ in a bank at the risk-free rate. Show there is an arbitrage opportunity. Give all actions at $t = T$, and discuss the type of arbitrage.

b) (559 only) If the call option price is instead $C_0 = $20, does the arbitrage opportunity still exist? What about $C_0 = $22? Discuss the type of arbitrage.
2. (10 points) (559 only) The value of an asset $S$ at time $T$ is modeled as a random variable $x$ with probability density

$$
\begin{align*}
    f &= 0 & f &= \frac{1}{5} & f &= 0 \\
    \text{for } x < 0, & \text{ for } 0 \leq x \leq 5, & \text{for } x > 5.
\end{align*}
$$

Call and put options with strike $K$ based on this asset have payoffs

$$
\Lambda_C(x) = \begin{cases} 
    x - K, & x \geq K \\
    0, & x < K
\end{cases}, \quad \Lambda_P(x) = \begin{cases} 
    0, & x \geq K \\
   K - x, & x < K
\end{cases}.
$$

a) Calculate $E[S(T)]$, the expected value of $S(T)$.

b) Find the expected payoff $E[\Lambda_C(S(T))]$. Consider cases $0 \leq K \leq 5$ and $K > 5$.

c) Show $\Lambda_P(x) - \Lambda_C(x) = K - x$, and use this identity with the results from a) and b) to find $E[\Lambda_P(S(T))]$. 

3. (10 points) (559 only) For an asset with value \( S \) at time \( t \), the Black-Scholes value for an option with payoff \( \Lambda(S(T)) \) is

\[
W(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{y=0}^{\infty} \Lambda(y)e^{-\frac{(\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}} \frac{dy}{y}.
\]

a) Let \( x = \frac{\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \) and show that

\[
W(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \Lambda(e^{x\sigma\sqrt{T-t} + \ln S + (r - \frac{\sigma^2}{2})(T-t)})e^{-\frac{x^2}{2}} dx.
\]

b) Use the formula from a) to calculate the Black-Scholes value for an asset-or-nothing call with payoff

\[
\Lambda(y) = \begin{cases} 
  y, & y \geq K \\
  0, & y < K 
\end{cases}
\]

Your answer should be written in terms of the function \( N(z) \) given on p. 1.
4. (10 points) a) Show that for the $d_1, d_2$ given on page 1, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$.

b) Assume the identity $SN'(d_1) - e^{-r(T-t)}KN'(d_2)$ is known. Show that for $C^{euro}(S,t)$ given on page 1,
$$\frac{\partial C^{euro}}{\partial S} = N(d_1).$$

c) A portfolio $\Pi$ consists of a unit of the asset $S$ and a quantity $k_0$ of call options $C^{euro}(S,t)$ such that $\frac{\partial \Pi}{\partial S}$, the delta of the portfolio, is zero at $t = 0$. Find $k_0$.

d) As $t \to T^-$, the call option out of the money. What is the delta of the portfolio as $t \to T^-$?