NAME (please print legibly): __________________________________________
Your University ID Number: _______________________________________

- Please try all questions. Questions are equally weighted. No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring in a formula sheet, written both sides with whatever they please. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Sharing of calculators is not allowed. Please show all work. Correct answers without supporting work may not receive full credit. You may use back pages if necessary.

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points) (459 only) The stock Xilinx had values as in the table below.

<table>
<thead>
<tr>
<th>Date</th>
<th>i</th>
<th>( S_i )</th>
<th>( S_i/S_{i-1} )</th>
<th>( U_i = \ln(S_i/S_{i-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>04/24/17</td>
<td>5</td>
<td>63.11</td>
<td>1.094</td>
<td>0.06</td>
</tr>
<tr>
<td>04/17/17</td>
<td>4</td>
<td>57.69</td>
<td>1.036</td>
<td>0.04</td>
</tr>
<tr>
<td>04/10/17</td>
<td>3</td>
<td>55.68</td>
<td>0.986</td>
<td>-0.01</td>
</tr>
<tr>
<td>04/03/17</td>
<td>2</td>
<td>56.48</td>
<td>0.976</td>
<td>-0.03</td>
</tr>
<tr>
<td>03/27/17</td>
<td>1</td>
<td>57.89</td>
<td>0.987</td>
<td>-0.01</td>
</tr>
<tr>
<td>03/20/17</td>
<td>0</td>
<td>58.63</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Calculate the mean \( \mu \) and standard deviation \( \sigma \) of the data \( U_i = \ln(S_i/S_{i-1}) \), \( i = 1..M \) and use the results to approximate the volatility of the asset \( S \).

Work starting with the given values \( U_i \): do NOT recompute these values.

\[
M = 5 \quad \mu = \frac{1}{M} \sum_{i=1}^{M} U_i = \frac{1}{5} (0.06 + 0.04 + 0.01 - 0.03 - 0.01) = \frac{1}{5} (0.05) = 0.01.
\]

\[
\sigma^2 = \frac{1}{M-1} \sum_{i=1}^{M} (U_i - \mu)^2 = \frac{1}{4} \left( (0.06 - 0.01)^2 + (0.04 - 0.01)^2 + (-0.01 - 0.01)^2 + (-0.03 - 0.01)^2 + (-0.01 - 0.01)^2 \right) = \frac{1}{4} \left[ (0.05)^2 + (0.03)^2 + (-0.02)^2 + (-0.04)^2 + (-0.02)^2 \right] = \frac{1}{4} \left[ 0.0025 + 0.0009 + 0.0004 + 0.0016 + 0.0004 \right] = \frac{1}{4} \left[ 0.0058 \right] = 0.00145; \quad \sigma \approx 0.0381.
\]

\[
\Delta t = \frac{1}{52}; \quad 6^2 \Delta t \approx \sigma^2; \quad 6 \approx \frac{\sigma}{\Delta t} = \frac{0.00145}{1/52}; \quad \sigma = \sqrt{\frac{0.00145}{1/52}} = 0.0754
\]

\[
6 \approx \frac{0.0381}{\sqrt{1/52}} \quad 6 \approx \sqrt{\frac{0.00145}{1/52}} \quad 6 \approx \sqrt{0.0754} \approx 0.275
\]
2. (10 points) Farmer George plans a crop of grapefruit that will be ready for harvest and sale as $N$ pounds of grapefruit juice in $T = 1$ year. The price per pound of grapefruit juice at $T$ is a random variable $G_T$, so the income $NG_T$ from the grapefruit juice is also a random variable.

a) Let $G_0$ be the current price per pound for grapefruit juice, let $r_G$ be the fractional annual return (return rate), and let stddev($r_G$) = $\sigma_G$, var($r_G$) = $\sigma_G^2$. Find $\text{var}(NG_T) = \text{cov}(NG_T, NG_T)$. Simplify.

$$\text{var}(NG_T) = \text{var}(N(G_0(1 + r_G))) = N^2G_0^2 \text{var}(1 + r_G)$$
$$= N^2G_0^2 \text{var} r_G = N^2G_0^2 \sigma_G^2.$$

b) There are no grapefruit juice futures contracts available, but there are orange juice futures. Each orange juice futures contract is for 1 pound of orange juice. The price per pound of orange juice at $T$ is a random variable $F_T$.

Let $F_0$ be the current price per pound for orange juice, let $r_F$ be the fractional annual return (return rate), and let stddev($r_F$) = $\sigma_F$, var($r_F$) = $\sigma_F^2$. In $T = 1$ year, each contract will be worth $F_T - F_0$.

George considers the hedged position $P_T(h) = NG_T - h(F_T - F_0)$ where $h$ is the number of orange juice futures contracts written. Let $\text{cov}(r_F, r_G) = \sigma_{FG}$.

Find $\text{var}(P_T(h)) = \text{cov}(NG_T - h(F_T - F_0), NG_T - h(F_T - F_0))$ as a quadratic in $h$. Simplify.

$$\text{var}(P_T) = \text{cov}(NG_T, NG_T) - 2h \text{cov}(NG_T, N(F_T - F_0))$$
$$- h^2 \text{cov}(N(F_T - F_0), (F_T - F_0))$$
$$= N^2 \text{cov}(G_T, G_T) - 2hN \text{cov}(G_T, F_T - F_0) + h^2 \text{cov}(F_T - F_0, F_T - F_0)$$

But $F_T = F_0(1 + r_F)$ so $F_T - F_0 = F_0 r_F$ and $G_T = G_0(1 + r_G)$

$$\therefore \text{var}(P_T) = N^2 \text{cov}(G_0(1 + r_G), G_0(1 + r_G))$$
$$- 2hN \text{cov}(G_0(1 + r_G), F_0 r_F) + h^2 \text{cov}(F_0(1 + r_F), F_0(1 + r_F))$$
$$= N^2G_0^2 \text{cov}(r_G, r_G) - 2hNG_0F_0 \text{cov}(r_G, r_F) + h^2F_0^2 \text{cov}(r_F, r_F)$$
$$= N^2G_0^2 G_0^2 - 2hNG_0F_0 GF_0 + h^2F_0^2 G_0^2.$$
c) For what value $h = h_{\text{min}}$ is the minimum variance of $P_T(h)$ attained?

$$\frac{d}{dh} \text{var}(P_T) = 0 - 2NG_0 F_0 \frac{G}{G_0} 6_{FG} + 2\alpha F_0^2 \frac{G}{G_0} 6_F^2$$

$h_{\text{min}}$ such that $\frac{d}{dh} \text{var}(P_T) = 0$

$$h_{\text{min}} = \frac{NG_0 F_0 \frac{G}{G_0} 6_{FG}}{F_0^2 \frac{G}{G_0} 6_F^2} = \frac{NG_0}{F_0} \frac{6_{FG}}{6_F^2}$$

\[ \text{\textit{Suggest}}. \]

d) (559 only) Calculate $\text{var}(P_T(h_{\text{min}}))$. Simplify.
3. (10 points) Recall the Black-Scholes PDE and Euro call payoff:

\[ \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2} - rC = 0, \quad C(S,T) = \begin{cases} 
S - K & \text{if } S > K \\
0 & \text{if } S \leq K
\end{cases} \]

a) Let \( f(S,t) \equiv \frac{\partial C}{\partial t}(S,t) \) where \( C(S,t) \) is the value of a Euro call.

Use the PDE and call payoff to find \( f(S(T),T) \) in the cases \( S(T) > K \) and \( S(T) < K \).

By PDE at \( t = T \), \( f(S,T) = \frac{2C}{S}S(S,T) = 2C(S,T) - \frac{S^2}{2}\frac{\partial^2 C}{\partial S^2}(S,T) \)

\[
= \begin{cases} 
\frac{\partial S(S - K) - \partial S^2}{\partial S^2}(S - K) - \frac{1}{2}\partial^2 S^2(S - K) & \text{if } S > K \\
\partial S(0) - \partial S^2(0) & \text{if } S < K
\end{cases}
\]

\[
f(S(T),T) = \begin{cases} 
\frac{S - K}{S} & \text{if } S(T) > K \\
0 & \text{if } S(T) < K
\end{cases}
\]

b) Show that \( f(S,t) \equiv \frac{\partial C}{\partial t}(S,t) \) satisfies the B-S PDE \( f_t + rSf_s + \frac{1}{2}\sigma^2S^2f_{ss} - rf = 0 \)

By PDE, \( C_t + rS C_s + \frac{1}{2}\sigma^2S^2 C_{ss} - rC = 0 \).

Take \( \frac{\partial}{\partial t} C : \)

\( C_t + rS C_s + \frac{1}{2}\sigma^2S^2 C_{ss} - rC_t = 0 \).

But \( f = C_t \), \( \Rightarrow C_t = f_t \), \( C_t = f_t \), \( C_{ss} = f_{ss} \), \( C_t = f_t \), \( C_{ss} = f_{ss} \):

\( f_t + rS f_s + \frac{1}{2}\sigma^2S^2 f_{ss} - rf = 0 \).

c) Given a) and b), make a reasonable guess about the relationship between the formula for \( f(S,t) \) and the formula for \( C^{\text{con}}(S,t;A) \), the Euro cash-or-nothing call that returns \( A \) if \( S(T) > K \), \( A/2 \) if \( S(T) = K \), 0 if \( S(T) < K \).

\( C^{\text{con}}(S(T),T;A) = \begin{cases} 
A & \text{if } S(T) > K \\
0 & \text{if } S(T) \leq K
\end{cases} \). Guess \( A = -\lambda K \).

\( C^{\text{con}}(S(t),T;A) = \begin{cases} 
S - K & \text{if } S(T) > K \\
0 & \text{if } S(T) \leq K
\end{cases} \). Guess \( f(S,t) = C^{\text{con}}(S,t;A) \).

d) The exact formulas are \( f(S,t) = SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} e^{-r(T-t)} N(d_2) \) and \( C^{\text{con}}(S,t;A) = A e^{-r(T-t)} N(d_2) \). Was the guess in c) correct? (Yes/No) extra term

\( (559 \text{ only}) \) Discuss the behavior of the term \( SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \).
4. (10 points) (459 only) The Black-Scholes formulas are

\[ C_{\text{euro}}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P_{\text{euro}}(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \] where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

a) Calculate \( \lim_{t \to 0^+} P_{\text{euro}}(S,t) \)

b) Calculate \( \lim_{S \to \infty} P_{\text{euro}}(S,t) \)

c) Calculate \( \lim_{S \to 0^+} P_{\text{euro}}(S,t) \)

d) Calculate \( \lim_{t \to T^-} P_{\text{euro}}(S,t) \)

\[ a) \text{as } t \to 0^+, \quad d_1 \to d_1(0) = \frac{\ln S + (r + \frac{1}{2}\sigma^2)T - \ln K}{\sigma \sqrt{T}} \]

\[ d_2 \to d_2(0) = \frac{\ln S + (r - \frac{1}{2}\sigma^2)T - \ln K}{\sigma \sqrt{T}} \]

\[ P_{\text{euro}} \to Ke^{-r(T-t)}N(-d_2(0)) - SN(-d_1(0)) \]

b) \( \text{as } S \to \infty, \quad d_1 \to \infty \text{ and } d_2 \to \infty, \quad -d_1 \to -\infty \text{ and } -d_2 \to -\infty \)

\[ P_{\text{euro}} \to 0 \]

\[ N(-d_1) \to 0 \text{ and } N(-d_2) \to 0 \]

c) \( \text{as } S \to 0^+, \quad \ln S \to -\infty \quad \text{so} \quad d_1 \to -\infty, \quad d_2 \to -\infty, \quad -d_1 \to +\infty, \quad -d_2 \to +\infty \)

\[ N(-d_1) \to 1 \text{ and } N(-d_2) \to 1 \]

\[ P_{\text{euro}} \to Ke^{-r(T-t)} \]

d) \( \text{as } t \to T^-, \quad \text{if } S > K \quad d_1 \to \infty, \quad d_2 \to \infty, \quad -d_1 \to -\infty, \quad -d_2 \to -\infty \)

\[ P_{\text{euro}} \to 0 \quad \text{if } S > K, \]

\[ N(-d_1) \to 0, \quad N(-d_2) \to 0 \]

\[ \text{if } S < K, \quad \text{if } S < K, \quad d_1 \to -\infty, \quad d_2 \to -\infty, \quad -d_1 \to +\infty, \quad -d_2 \to +\infty \]

\[ P_{\text{euro}} \to Ke^{-rT} - S = K - S, \]

\[ N(-d_1) \to 1, \quad N(-d_2) \to 1 \]
5. (10 points) You hold a Bermudan style call option with strike $K$. The Bermudan option may be exercised at $t = t_0 \equiv T/2$ or at $t = T$, but not at any other time $t$.

a) Suppose $S(t_0) > K$. If you exercise the option at $t_0$, what is your profit or loss?

b) Suppose $S(t_0) > K$. You short one unit of the asset, receiving $S(t_0)$ in cash. You deposit the amount $K$ in a bank at the risk-free rate $r$. At time $T$, you withdraw all the resulting cash from the bank. If $S(T) > K$, you exercise the call option to purchase the asset for $K$. If $S(T) \leq K$, you purchase the asset on the spot market for $S(T)$. You return the asset to the owner.

Using this strategy, what is your profit or loss? Express the result in terms of cash at $t_0$, using the discount factor $e^{-r(T-t_0)}$ to convert cash at $t = T$ to cash at $t_0$.

c) Show that the scheme described in b) is more profitable than the scheme in a). Show that in all cases $S(t_0) > K$, $S(t_0) = K$, $S(t_0) < K$ it does not pay to exercise a Bermudan call early. It follows that $C^{\text{berm}}(S,t;K) = C^{\text{euro}}(S,t;K)$.

d) (559 only) Show that $P^{\text{berm}}(S,t;K) \geq P^{\text{euro}}(S,t;K)$ for $0 < t \leq T$ and that $P^{\text{berm}}(S,t;K) = P^{\text{euro}}(S,t;K)$ for $t_0 < t \leq T$.

a) Since $S(t_0) > K$, call option payoff if exercised at $t_0$

$$\max(S(t_0) - K, 0) = S(t_0) - K > 0.$$  

b) Shorting asset generates income of $-S(t_0)$ at $t_0$.

Of this, deposit $K$ in bank, so net cash at $t$ is $S(t) - K$.

At $t = T$, withdraw $Ke^{r(T-t)}$ from bank.

If $S(T) > K$, exercise call option to purchase asset for $K$.
If $S(T) < K$, purchase asset on spot market for $S(T)$.

Net cash at $t = T$ in \begin{cases} \hfill Ke^{r(T-t)} - K \quad \text{if } S(T) > K, \\ K e^{r(T-t)} - S(T) \quad \text{if } S(T) \leq K. \end{cases}

Since $e^{r(T-t)} > 1$, $Ke^{r(T-t)} - K > 0$.

Also, if $S(T) \leq K$, $-S(t) > -K$ and $Ke^{r(T-t)} - S(t) > Ke^{r(T-t)} - K > 0$.

In either case $S(T) > K$ or $S(T) < K$, the net cash at $t = T$ is positive.
c) In the case $S(t_0) > K$,

Since the net cash at $t = T$ is positive in scheme b), so is its value at $t = t_0$:

exercising the call at $t = t_0$ as in a) gives $S(t_0) - K$,

but the scheme b) gives this same amount plus something positive: scheme b) is more profitable.

The comparison of schemes a) and b) shows that in the case $S(t_0) > K$, it does not pay to exercise early.

Two cases remain: $S(t_0) < K$ and $S(t_0) = K$.

If $S(t_0) \leq K$, one would not exercise the call option because doing so would be paying $K$ to acquire an asset that can be purchased for only $S(t_0)$ on the spot market.
1) at times \(0 < t < t_0\),

since the Bermudan put option can be exercised early (at \(t = t_0\)), this extra flexibility makes

\[
P_{\text{Berm}}(s, t; K) \geq P_{\text{Euro}}(s, t; K).
\]

At times \(t_0 < t < T\)

there is no chance to exercise the Bermudan put early; it must be exercised at \(t = T\), the same as the Euro put option. It follows

\[
P_{\text{Berm}}(s, t; K) = P_{\text{Euro}}(s, t; K)
\]

By continuity, \(P_{\text{Berm}}(s, t_0; K) = P_{\text{Euro}}(s, t_0; K)\).

Then at all times \(0 < t < T_0\),

\[
P_{\text{Berm}}(s, t; K) \geq P_{\text{Euro}}(s, t; K).
\]
6. (10 points) The three figures show an asset and the corresponding Euro call and put options, both with the same expiration $T = 1$ and strike $K$. At $t = t_d = \frac{1}{2}$ a dividend is paid.

a) Which figure shows the asset, and why?

*Figure B is the asset because it jumps down at $t = t_d$ when dividend is paid.*

b) Which figure shows the call option, and why?

*Figure A shows call option because it tends to increase over time intervals when the asset increases.*

c) Which figure shows the put option, and why?

*Figure C, because it decreases over time intervals when the asset increases.*
d) Find the strike $K$ from the figure(s).
Hint: Look at the values of the asset and of the in-the-money option at expiration.

The asset has $S(T) \approx 0.98$
the put option fig. C is "in the money" $P(T) \approx 0.02$

Since for put, payoff is $\max(K - S, 0)$,
for put "in the money", $K - S(T) = P(T)$

$$K = S(T) + P(T) = 0.98 + 0.02 = 1.$$ __e) Find the dividend yield $d_y$ from the figure for the asset.
Suppose a dividend with this same dividend yield $d_y$ is paid every $\frac{1}{2}$ year. That is, there are dividend payments at $t_d = 0.5$, $t_d = 1$, $t_d = 1.5$, etc. If we model these discrete dividend payments by a continuously paid dividend with dividend rate $q$, what is the value of $q$?

$d_y$ is such that $S(t_d^+) = (1 - d_y) S(t_d^-)$:

$S(t_d^+) \approx 1.07$, $S(t_d^-) \approx 0.97$. Using these values,

$$0.97 = (1 - d_y) 1.07$$

$$1 - d_y = \frac{0.97}{1.07} \approx 0.9; \quad d_y \approx 0.1$$

A continuously paid dividend with rate $q$ over $at = \frac{1}{2}$ year, would cause the asset to decrease by $e^{-atq} = e^{-q}$.

This factor should roughly equal $1 - d_y$.

$$e^{-q} = 1 - d_y$$

$$-\frac{q}{2} = \ln(1 - d_y) = \ln(0.9) \approx -0.1$$

$$q = -2 \ln(0.9) \approx 1.2 \text{ or } 20\%.$$


7. (10 points) a) Rank the following options in order of increasing value:

i) Asian put with payoff \( \max(K - S_{\text{avg}}, 0) \) where \( S_{\text{avg}} = \frac{1}{T} \int_{0}^{T} S(t) dt \),

ii) Lookback put with payoff \( \max(K - S_{\text{max}}, 0) \) where \( S_{\text{max}} = \max_{0 \leq t \leq T} S(t) \),

iii) Lookback put with payoff \( \max(K - S_{\text{min}}, 0) \) where \( S_{\text{min}} = \min_{0 \leq t \leq T} S(t) \).

Since \( S_{\text{min}} \leq S(t) \leq S_{\text{max}} \), taking averages,

\[
\frac{1}{T} \int_{0}^{T} S_{\text{min}} dt \leq \frac{1}{T} \int_{0}^{T} S(t) dt \leq \frac{1}{T} \int_{0}^{T} S_{\text{max}} dt
\]

\( S_{\text{min}} \leq S_{\text{avg}} \leq S_{\text{max}} \).

i) \( K - S_{\text{min}} \geq K - S_{\text{avg}} \geq K - S_{\text{max}} \)

\[
\max(K - S_{\text{min}}, 0) \geq \max(K - S_{\text{avg}}, 0) \geq \max(K - S_{\text{max}}, 0)
\]

\( \text{i) } \geq \text{ ii) } \geq \text{ iii) } \)

b) Draw a sketch of the region \( 0 \leq t \leq T, S \geq 0 \) of the \((S, t)\) plane. Suppose \( 0 < B_1 < B_2 < K \) and show the barriers \( B_1, B_2 \) as line segments. Label the point \((S, t) = (K, T)\) with the letter \( K \). Include various asset trajectories, all starting at the same point \((S_0, 0)\) with \( S_0 > B_2 \). Include asset trajectories that cross the barriers as well as trajectories that end up above and below the strike without crossing either barrier.

![Diagram of asset trajectories](image)

c) Which is worth more: a Euro down-and-out call with strike \( K \) and barrier \( B_1 \), or a Euro down-and-out call with strike \( K \) and barrier \( B_2 \)?

The call with the barrier \( B_1 \) is worth more, because
8. (10 points) (459 only)

Suppose \( f(x) \) is given, and \( X \) is a random variable from the uniform distribution on \([-1, 1]\).

a) Basic Monte-Carlo uses \( E[f(X)] \approx \frac{1}{M} \sum_{i=1}^{M} f(X_i) \) and antithetic Monte-Carlo computation uses \( E[f(X)] = E[f_0(X)] \approx \frac{1}{M} \sum_{i=1}^{M} f_0(X_i), \ f_0(x) = \frac{1}{2} (f(x) + f(-x)) \).

Give approximate 95% confidence intervals for these approximations. Write the standard deviations as \( \text{stddev}(f(X)) = \sigma \), \( \text{stddev}(f_0(X)) = \sigma_0 \).

Let \( \bar{f} = E[f_0(X)] \). For M.C., 95% c.i. is \( \bar{f} \pm 1.96 \frac{\sigma}{\sqrt{M}} \) and approximate 95% c.i. is \( \bar{a}_M \pm 1.96 \frac{\hat{b}_M}{\sqrt{M}} \) where \( \bar{a}_M = \frac{1}{M} \sum_{i=1}^{M} f_0(X_i) \), \( \hat{b}_M^2 = \frac{1}{M-1} \sum_{i=1}^{M} (f_0(X_i) - \bar{a}_M)^2 \).

For antithetic M.C., \( E[f_0(X)] = \bar{f}, \) 95% c.i. is \( \bar{f} \pm 1.96 \frac{\sigma_0}{\sqrt{M}} \) and approximate 95% c.i. is \( \bar{a}_0^0 \pm 1.96 \frac{\hat{b}_0^0}{\sqrt{M}} \) where \( \bar{a}_0^0 = \frac{1}{M} \sum_{i=1}^{M} f_0(X_i), \ f_0(x) = \frac{1}{2} (f(x) + f(-x)) \), \( (\hat{b}_0^0)^2 = \frac{1}{M-1} \sum_{i=1}^{M} (f_0(X_i) - \bar{a}_0^0)^2 \).

b) Suppose \( f \) is of the form \( f = a + bx \) for positive constants \( a, b \). Compute the confidence intervals from part c). Explain any unexpected results.

For \( f = a + bx \), \( \bar{a}_M = \frac{1}{M} \sum_{i=1}^{M} a + bx_i = a + b \bar{x} \) where \( \bar{x} = \frac{1}{M} \sum_{i=1}^{M} x_i \). Basic M.C.

\( \hat{b}_M^2 = \frac{1}{M-1} \sum_{i=1}^{M} (x_i - \bar{x})^2 \) \( \bar{b}_M = \sqrt{\frac{\hat{b}_M^2}{M-1}} \) approx 95% c.i. is \( a + b \bar{x} \pm 1.96 \frac{\hat{b}_M}{\sqrt{M}} \) where \( \hat{b}_M = \sqrt{\frac{1}{M-1} \sum_{i=1}^{M} (x_i - \bar{x})^2} \).

For antithetic M.C., \( f_0(x) = (f(x) + f(-x))/2 = (a + bx + a - bx)/2 = a = \text{constant} \). \( \bar{a}_0^0 = \frac{1}{M} \sum_{i=1}^{M} f_0(x_i) = a \), \( (\hat{b}_0^0)^2 = \frac{1}{M-1} \sum_{i=1}^{M} (f_0(x_i) - \bar{a}_0^0)^2 = \frac{1}{M-1} \sum_{i=1}^{M} (a - a)^2 = 0 \) approx 95% c.i. is \( a \pm 1.96 \frac{\hat{b}_0^0}{\sqrt{M}} = a \pm 0 \): unexpected.

In this example, since \( f_0 \) is constant, the antithetic M.C. scheme gives the exact result \( E[f_0] = a \) when applied with any \( M > 0 \).