No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

The Black-Scholes formulas are

\[ C_{\text{euro}}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P_{\text{euro}}(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where

\[
\begin{align*}
d_1 &= \frac{\ln S + (r + \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}}, \\
d_2 &= \frac{\ln S + (r - \frac{1}{2} \sigma^2)(T-t) - \ln K}{\sigma \sqrt{T-t}} \\
N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx
\end{align*}
\]

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1. (10 points) An asset has value \( S_0 = $120 \) at \( t = 0 \), and will have unknown value \( S(T) \) at time \( T \). A call option with strike \( K = $100 \) is currently \( (t = 0) \) selling for \( C_0 = $21 \), and the discount factor for the period 0 to \( T \) is \( e^{-rT} = 0.99 = 99/100 \). The risk-free rate is \( r \).

a) Consider the portfolio obtained by selling (short) one unit of the asset at \( t = 0 \), buying one call option, and investing the present value of the strike \( K \) in a bank at the risk-free rate. Show there is an arbitrage opportunity. Give all actions at \( t = T \), and discuss the type of arbitrage.

b) (559 only) If the call option price is instead \( C_0 = $20 \), does the arbitrage opportunity still exist? What about \( C_0 = $22 \)? Discuss the type of arbitrage.

\[ a) \quad \begin{align*}
@ t=0 & : & \text{borrow asset } S_0 = $120 \text{ sell for } S_0 = $120 \\
& & \text{buy call option for } C_0 = $21 \\
& & \text{invest } $100 e^{-rT} = $99 \text{ in bank}
\end{align*} \]

\[ @ t=1 & : & \text{remove } $100 \text{ from bank} \\
& & \text{if } S_T > 100 \text{ exercise call} \\
& & \text{option (i.e. buy asset for 100) and return it to owner} \\
& & \text{if } S_T < 100 \text{ do not exercise call buy asset at } S_T \text{ and return it to owner}
\]

\[ \text{NO PROFIT}\]

\[ \text{Profit} = 100 - S_T \]

This is \textbf{TYPE 2} Arbitrage because the position has zero initial cost but positive \( \checkmark \).
2. (10 points) (459 only) The value of an asset $S$ at time $T$ is modeled as a random variable $x$ with probability density

\[ f = 0 \quad \text{for} \quad x < 0, \quad f = \frac{1}{5} \quad \text{for} \quad 0 \leq x \leq 5, \quad f = 0 \quad \text{for} \quad x > 5. \]

Call and put options with strike $K$ based on this asset have payoffs

\[ \Lambda^C(x) = \begin{cases} 
 x - K, & x \geq K \\
 0, & x < K 
\end{cases} \quad \Lambda^P(x) = \begin{cases} 
 0, & x \geq K \\
 K - x, & x < K 
\end{cases} \]

a) Calculate $E[S(T)]$, the expected value of $S(T)$.

b) Find the expected payoff $E[\Lambda^C(S(T))]$ for $K = 3$.

c) Show $\Lambda^P(x) - \Lambda^C(x) = K - x$, and use this identity with the results from a) and b) to find $E[\Lambda^P(S(T))]$ for $K = 3$.

\[ E[S(T)] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_0^5 \frac{x}{5} \, dx = \frac{x^2}{10} \bigg|_0^5 = \frac{25}{10} = 2.5 \]

\[ E[\Lambda^C(S(T))] = E[\max(x - K, 0)] = \int_{-\infty}^5 \max(x - K, 0) f(x) \, dx = \int_3^5 \frac{x - 3}{15} \, dx = \frac{x^2}{10} \bigg|_3^5 = \frac{25}{10} - \frac{9}{10} + \frac{9}{5} = \frac{2}{5} = 0.4 \]

\[ E[\Lambda^P(S(T))] = \frac{2}{5} = 0.4 \]

C) ON BARK!
c) Show $\Lambda^0 - \Lambda^c = k - x$. Find $E[\Lambda^0(s|\alpha)]$ when $k = 3$

$\Lambda^c = \begin{cases} x-k & \text{if } x \geq k \\ 0 & \text{if } x < k \end{cases}$

$\Lambda^0 = \begin{cases} 0 & \text{if } x \geq k \\ k-x & \text{if } x < k \end{cases}$

$\Lambda^0 - \Lambda^c = \begin{cases} 0-x+k & \text{if } x \geq k \\ k-x-0 & \text{if } x < k \end{cases}$

Since both cases are the same,

$\Lambda^0 - \Lambda^c = k - x$.


$E[\Lambda^0(s|\alpha)] = 3 - 2.5 + .4 = .9$.

$\therefore$ 4/4
3. (10 points) (459 only) For an asset with value $S$ at time $t$, the Black-Scholes value for an option with payoff $\Lambda(S(T))$ is

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}\sigma^2(T-t)} \int_{y=0}^{\infty} \Lambda(y)e^{-\frac{(\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}}\frac{dy}{y}$$

Assume that the change of variable $x = \frac{\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$ gives

$$W(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \Lambda(e^{\sigma \sqrt{T-t} + \ln S + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{x^2}{2}} dx$$

a) Show that if $y \leq K$, then $x \leq -d_2$ and if $y \geq K$, then $x \geq -d_2$, where $d_2$ is on page 1.

b) Use the result from a) to calculate the Black-Scholes value for a cash-or-nothing put with payoff

$$\Lambda(y) = \begin{cases} A, & y \leq K \\ 0, & y > K \end{cases}$$

Your answer should be written in terms of the function $N(z)$ given on p. 1.

a) If $y < K$ then $\ln y < \ln K$,

$$x = \frac{\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)^2}{\sigma^2(T-t)}$$

so $x \leq -d_2$

If $y \geq K$ then $\ln y \geq \ln K$,

$$x = \frac{\ln y - \ln S - (r - \frac{\sigma^2}{2})(T-t)^2}{\sigma^2(T-t)}$$

so $x \geq -d_2$

b) On back!
b) \[ P_{\text{cas}h}(S_T) = \frac{e^{-r(T-t)}}{\sqrt{2\pi \sigma^2(T-t)}} \int_{y=0}^{K} A \exp \left[ -\frac{\ln(y/K) + \frac{(y-K)^2}{2\sigma^2(T-t)}}{2\sigma^2(T-t)} \right] \frac{dy}{y} \]

\[ = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} A \cdot y^{1/2} \, dy = A e^{-r(T-t)} \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} y^{1/2} \, dy\right) \]

\[ P_{\text{cas}h} = A e^{-r(T-t)} \ast N(-d_2) \]
4. (10 points) a) Show that for the $d_1, d_2$ given on page 1, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$.

b) Assume the identity $SN'(d_1) - e^{-r(T-t)} KN'(d_2) = 0$ is known. Show that for $C_{\text{euro}}(S,t)$ given on page 1,

$$\frac{\partial C_{\text{euro}}}{\partial S} = N(d_1).$$

c) A portfolio $\Pi$ consists of a unit of the asset $S$ and a quantity $k_0$ of call options $C_{\text{euro}}(S,t)$ such that $\frac{\partial \Pi}{\partial S}$, the delta of the portfolio, is zero at $t = 0$. Find $k_0$.

d) As $t \to T^-$, the call option out of the money. What is the delta of the portfolio as $t \to T^-$?

\[
\frac{\partial d_1}{\partial S} = \frac{1}{\sigma \sqrt{T-t}} \frac{\partial}{\partial S} \left( \ln S + (r - \frac{\sigma^2}{2})(T-t) - \ln K \right) \\
= \frac{1}{\sigma \sqrt{T-t}} x \left( \frac{1}{S} + O \right)
\]

\[
\frac{\partial d_2}{\partial S} = \frac{1}{\sigma \sqrt{T-t}} \frac{\partial}{\partial S} \left( \ln S + (r + \frac{\sigma^2}{2})(T-t) - \ln K \right) \\
= \frac{1}{\sigma \sqrt{T-t}} x \left( \frac{1}{S} + O \right)
\]

\[
\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} \checkmark \quad a) \ 2/2
\]

b) $\frac{\partial C_{\text{euro}}}{\partial S} = \frac{\partial}{\partial S} \left[ SN(d_1) - Ke^{-r(T-t)} N(d_2) \right]$

$$= N(d_1) \frac{\partial S}{\partial S} + S N(d_1) \frac{\partial}{\partial S} N(d_1) - Ke^{-r(T-t)} N(d_2) \frac{\partial}{\partial S} N(d_2)$$
b) \[ \frac{dC}{ds} = N(d_i) + \frac{\partial C}{\partial s} N'(d_i) \frac{d}{ds} k e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial s} \]

\[ \frac{2d_1}{ds} = \frac{\partial d_2}{\partial s} \]

\[ \frac{2C}{ds} = N(d_i) + [SN'(d_i) - k e^{-r(T-t)} N'(d_2)] \frac{\partial d_2}{\partial s} = 0 \]

\[ c) \frac{P}{\tau} = k_0 C + S \quad \frac{\partial P}{\partial s} = k_0 \frac{\partial C}{\partial s} + 1 \]

\[ \frac{\partial P}{\partial s} = k_0 N(d_i) + 1 = 0 \]

\[ k_0 = -\frac{1}{N(d_i)} \quad \text{at } t=0 \]

\[ d) \text{for } d_1, \text{ if option out of the money, } \]

\[ \text{numerator } < 0 \quad \text{denominator } \to 0 \quad d_1 \to -\infty \]

\[ \text{As } d_1 \to -\infty \quad N(d_i) \to 0 \]

\[ \text{so } \frac{\partial P}{\partial s} \to 1 \]