NAME (please print legibly): __________________________________________________________
Your University ID Number: ________________________________________________________

- No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

The Black-Scholes formulas are

\[ C_{\text{euro}}(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad P_{\text{euro}}(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \]

where

\[ d_1 = \frac{\ln S + (r + \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln S + (r - \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma\sqrt{T-t}} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \]

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1. **(10 points)** An asset has value $S_0 = $120 at $t = 0$, and will have unknown value $S(T)$ at time $T$. A call option with strike $K = $100 is currently ($t = 0$) selling for $C_0 = $21, and the discount factor for the period 0 to $T$ is $e^{-rT} = 0.99 = 99/100$. The risk-free rate is $r$.

a) Consider the portfolio obtained by selling (short) one unit of the asset at $t = 0$, buying one call option, and investing the present value of the strike $K$ in a bank at the risk-free rate. Show there is an arbitrage opportunity. Give all actions at $t = T$, and discuss the type of arbitrage.

b) (559 only) If the call option price is instead $C_0 = $20, does the arbitrage opportunity still exist? What about $C_0 = $22? Discuss the type of arbitrage.
2. (10 points) (459 only) The value of an asset $S$ at time $T$ is modeled as a random variable $x$ with probability density

\[ f(x) = \begin{cases} 
0 & \text{for } x < 0, \\
\frac{1}{5} & \text{for } 0 \leq x \leq 5, \\
0 & \text{for } x > 5.
\end{cases} \]

Call and put options with strike $K$ based on this asset have payoffs

\[ \Lambda^C(x) = \begin{cases} 
x - K, & x \geq K \\
0, & x < K
\end{cases}, \quad \Lambda^P(x) = \begin{cases} 
0, & x \geq K \\
K - x, & x < K
\end{cases}. \]

a) Calculate $E[S(T)]$, the expected value of $S(T)$.

b) Find the expected payoff $E[\Lambda^C(S(T))]$ for $K = 3$.

c) Show $\Lambda^P(x) - \Lambda^C(x) = K - x$, and use this identity with the results from a) and b) to find $E[\Lambda^P(S(T))]$ for $K = 3$. 
3. (10 points) (459 only) For an asset with value $S$ at time $t$, the Black-Scholes value for an option with payoff $\Lambda(S(T))$ is

$$W(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi \sigma^2(T-t)}} \int_{y=0}^{\infty} \Lambda(y) e^{-\frac{(\ln y - \ln S - (r - \sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}} \frac{dy}{y}$$

Assume that the change of variable $x = \frac{\ln y - \ln S - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{(T-t)}}$ gives

$$W(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \Lambda(e^{x\sigma \sqrt{T-t}+\ln S+(r-\sigma^2/2)(T-t)}) e^{-\frac{x^2}{2}} dx$$

a) Show that if $y \leq K$, then $x \leq -d_2$ and if $y \geq K$, then $x \geq -d_2$, where $d_2$ is on page 1.

b) Use the result from a) to calculate the Black-Scholes value for a cash-or-nothing put with payoff

$$\Lambda(y) = \begin{cases} A, & y \leq K \\ 0, & y > K \end{cases}$$

Your answer should be written in terms of the function $N(z)$ given on p. 1.
4. **(10 points)** a) Show that for the $d_1, d_2$ given on page 1, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$.

b) Assume the identity $SN'(d_1) - e^{-r(T-t)}KN'(d_2)$ is known. Show that for $C_{\text{euro}}(S, t)$ given on page 1,

$$\frac{\partial C_{\text{euro}}}{\partial S} = N(d_1).$$

c) A portfolio $\Pi$ consists of a unit of the asset $S$ and a quantity $k_0$ of call options $C_{\text{euro}}(S, t)$ such that $\frac{\partial \Pi}{\partial S}$, the delta of the portfolio, is zero at $t = 0$. Find $k_0$.

d) As $t \to T^-$, the call option out of the money. What is the delta of the portfolio as $t \to T^-$?