For random variables $X$, $Y$ and constants $a$, $b$, $c$, $d$:

$$\text{Var}(X) = E[(X - \bar{X})^2], \quad \text{Cov}(X, Y) = E[(X - \bar{X})(Y - \bar{Y})]$$

$$\text{Var}(X + Y) = \text{Var}(X) + 2 \text{Cov}(X, Y) + \text{Var}(Y)$$

$$\text{Var}(X + a) = \text{Var}(X), \quad \text{Cov}(X + a, Y + b) = \text{Cov}(X, Y)$$

$$\text{Var}(cX) = c^2 \text{Var}(X), \quad \text{Cov}(cX, dY) = cd \text{Cov}(X, Y)$$

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1. (10 points) A venture capitalist uses the utility \( U(x) = \sqrt{x} \) (where \( x \) is in millions of dollars) to evaluate two possible investments (i) or (ii) for the coming year.

(i) Buy Treasury bills, which will give her a wealth of 11M for sure (M = million dollars.)

(ii) Make a risky investment called \( X \), which has three possible outcomes:

\[
X = \begin{cases} 
16M & \text{with probability 0.4} \\
9M & \text{with probability 0.5} \\
4M & \text{with probability 0.1}
\end{cases}
\]

a) Based upon expected utility, which alternative (i) or (ii) should she choose? Explain.

b) What is the certainty equivalent of alternative (ii)?

c) Suppose the utility function is \( V(x) = 4\sqrt{x} - 3 \). Would using \( V(x) \) in place of \( U(x) \) change the results in parts a) and b)?

d) (559 only) Suppose the utility function is \( W(x) = x \). Would using \( W(x) \) in place of \( U(x) \) change the results in parts a) and b)?

A) Payoff of (ii) is \( \sqrt{11M} = 3.3166M \) dollars

\[
E(U(x)) = 0.4 \times 16 + 0.5 \times 9 + 0.1 \times 4 \\
= 0.4 \times 16 + 0.5 \times 9 + 0.1 \times 4 \\
= 3.3 \text{ M dollars}
\]

\( \sqrt{11} > 3.3 \), so choose (i)

b) Certainty equivalent \( C = (3.3) = 10.89 \) (M. dollars)

c) \( V(x) = 4\sqrt{x} - 3 \), \( W(x) = x \), then \( V(x) \) and \( W(x) \) are equivalent utility functions, so result doesn't change.

d) \( W(11) = 11 \), \( E(W(x)) = 0.4 \times 1 + 0.5 \times 9 + 0.1 \times 4 = 11.3 \).

11 < 11.3, then in this case prefer (iii) rather than (i)

E. Certainty equivalent is just 11.3.
2. (10 points) A bond is being considered as possible investment. The bond has three years to run, and in three years will pay the face value of 10,000 dollars. The bond pays yearly coupons of c dollars for some c ≥ 0. A coupon has just been paid to the current owner, so if you purchase the bond your first coupon will be one year from now.

Suppose the current term structure of interest rates is:

\[(s_1, s_2, s_3) = (0.05, 0.06, 0.07)\]

The first rate is for funds to be lent for one year, starting now. The second rate is for funds to be lent for a period of two years, starting now, compounded annually. The third rate is for funds to be lent for a period of three years, starting now, compounded annually.

Let \(d_{0,j}\) be the discount factor for the period from now to \(j\) years from now: cash received \(j\) years from now is multiplied by \(d_{0,j}\) to determine the present value of the cash. The discount factors are determined by the term structure of interest rates given above.

a) Fill in the blanks of the cash-flow diagram on the next page, for the cash stream of the bond. Use the symbol \(c\) for the coupon payout, and write \(S(c)\) for the amount to be paid for the bond. Use approximate values \(\frac{1}{1.05} = 0.9524, \frac{1}{1.06^2} = 0.8900, \frac{1}{1.07^3} = 0.8163\)

b) Find the formula for \(S(c)\)

c) Sketch \(S(c)\) for \(0 ≤ c ≤ 1000\).

\[S(c) = F( d(0,3) ) + c(k)(d(0,3)) + c(k)(d(1,3)) + c(k)(d(2,3))\]

\[F = 10,000\] \[c(k) = c\] \[d(0,3) = 0.8163\] \[d(1,3) = 0.890\] \[d(2,3) = 0.9524\]

\[S(c) = 10,000(0.8163) + c(0.8163) + c(0.890) + c(0.9524)\]

\[S(c) = 8163 + 2.6587c\]

When \(c = 0\), \(S(c) = 8163\)

When \(c = 1000\), \(S(c) = 10821.7\)
\[ \Delta t = \text{1 year} \]

Cash at times \( t_j = j \Delta t \)

- \( -S(C) \)
  
  \( +C \)  \( +C \)  \( +C \)  \( 10,000 \)

Discount factors \( d_{0,j} \)

\[ d_{0,1} = 0.9524 \]

\[ d_{0,2} = 0.8900 \]

\[ d_{0,3} = 0.8163 \]
3. (10 points) A company has a $10 million portfolio $P$ with a beta of $\beta_p = 1.6$ relative to the market $M$. The market is represented by the Standard and Poors index $M$, currently 2000, and the company would like to use S&P futures to hedge. Each futures contract is for delivery of $50$ times the S&P index, so $F(T) = 50M(T)$. Assume without showing

$$\frac{\text{Cov}(P(T), M(T))}{\text{Var}(M(T))} = \frac{P(0)}{M(0)}\beta_p.$$

a) Write the formula for $y(h)$, the value at $T$ of the hedged portfolio consisting of $P$ together with the gain (or loss) from selling $h$ futures contracts. Ignore interest, so the discount factor from the present until delivery is 1.

b) Write the formula for $\text{Var}(y(h))$ as a sum of three terms each with a different power of $h$ (see formulas on page 1.)

c) What is the number $\tilde{h}$ of contracts that should be sold to minimize $\text{Var}(y(h))$?

d) Make a rough sketch of $\text{Var}(y(h))$ for $h$ near $\tilde{h}$. Label the horizontal axis as $h$ and the vertical axis as $\text{Var}(y(h))$. Label the point $(\tilde{h}, \text{Var}(y(\tilde{h})))$. When doing the sketch, assume that $\text{Var}(y(\tilde{h})) > 0$.

e) (559 only) Discuss the case $\text{Var}(y(\tilde{h})) = 0$.

\[ a) \ \frac{1}{2} \]

\[ b) \ \text{Var}(y(h)) = \text{Var}(P(T)) + h^2 \text{Var}(50M(T) - F(0)) + 2h \text{Cov}(P(T), 50M(T) - F(0)) \]

\[ = \text{Var}(P(T)) + 50^2 h^2 \text{Var}(M(T)) + 2(50h) \text{Cov}(P(T), M(T)) \]

\[ c) \ \frac{\text{dVar}(y(h))}{\text{dh}} = 0 + 8100 \text{Cov}(P(T), M(T)) + 2(50^2) h \text{Var}(M(T)) = 0 \]

\[ h = -\frac{8100}{50^2 \text{Var}(M(T))} = -\frac{8100}{50 \text{Var}(M(T))} \]

\[ = -\frac{1}{50} \cdot \frac{\beta_p}{\text{Var}(M(T))} = -\frac{1}{50} \cdot \frac{(10 \cdot 16000)}{50} = -160 \]

e) $\frac{3}{3}$

\[ \text{Sell} 160 \text{ contracts to minimize } \text{Var}(y(h)). \]

\[ \text{Variance is minimized near } \tilde{h}. \]

\[ l) \ \frac{3}{3} \]