NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

• No books, collaboration or access to outside material is allowed, with two exceptions. Each student may bring one sheet of notes. Each student may bring in a calculator such as the TI-83 Plus and the TI-84 Plus family, allowed for use on the PSAT, the SAT Subject Tests, Math Level 1 and 2 Tests, AP Calculus exam and ACT Test. Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

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1. (10 points) A venture capitalist uses the utility $U(x) = x^{1/3}$ (where $x$ is in millions of dollars) to evaluate two possible investments (i) or (ii) for the coming year.

(i) Buy Treasury bills, which will give her a wealth of $10M$ ($M = \text{million dollars}$) with certainty (risk-free).

(ii) Make a risky investment $X$ which has three possible outcomes:

$$X = \begin{cases} 
27M & \text{with probability 0.2} \\
8M & \text{with probability 0.7} \\
1M & \text{with probability 0.1}
\end{cases}$$

a) Based upon expected utility, which alternative (i) or (ii) should she choose? Explain.

b) What is the certainty equivalent of alternative (ii)?

c) Suppose the utility function is $W(x) = x$. Does using $W(x)$ in place of $U(x)$ change the result in part a)? Explain.

(a) (i) $U = \sqrt[3]{10} = 2.1544$

(ii) $E(U) = 0.2\sqrt[3]{27} + 0.7\sqrt[3]{8} + 0.1\sqrt[3]{1}$

$= 0.2(3) + 0.7(2) + 0.1(1) = 2.1$

[choose (i)] because $10^{1/3} > 2.1$

(b) $U(c) = 2.1$

$c^{1/3} = 2.1$

$c = 9.261$

(c) (i) $W = 10$

(ii) $E(W) = 0.2(27) + 0.7(8) + 0.1(1) = 11.1$

[yes] it does. (ii) is now preferable.
2. (10 points) A bond is being considered as possible investment. The bond has two years to run, and in two years will pay the face value of 10,000 dollars. The bond pays yearly coupons of 500 dollars. A coupon has just been paid to the current owner, so if you purchase the bond your first coupon will be one year from now.

Suppose the current term structure of interest rates is: \((s_1, s_2) = (0.04, 0.05)\). The first rate is for funds to be lent for one year, starting now. The second rate is for funds to be lent for a period of two years, starting now, compounded annually.

Let \(d_{0,j}\) be the discount factor for the period from now to \(j\) years from now: cash received \(j\) years from now is multiplied by \(d_{0,j}\) to determine the present value of the cash. The discount factors \(d_{0,1}\) and \(d_{0,2}\) are determined by the term structure of interest rates given above.

a) Fill in the blanks of the cash-flow diagram on the next page, for the cash stream of the bond. Use approximate values \(\frac{1}{1.04} = 0.9615\), \(\frac{1}{1.05^2} = 0.9070\) and write \(S\) for the amount to be paid for the bond.

b) Solve for \(S\).

c) Suppose that instead of the values given above, the rates are equal: \(s_1 = s_2 = r\). Find an equation \(F(r) = 0\) for the rate \(r\) which would result in the same value \(S\) for the bond.

\[
b) \quad S = 500 \cdot 0.9615 + 500 \cdot 0.9070 + 10,000 \cdot 0.9070
\]

\[
S = 10,040.25
\]

\[
c) \quad F(r) = -10,004.25 + \frac{500}{(1+r)} + \frac{500}{(1+r)^2} + \frac{10,000}{(1+r)^2}
\]
\( \Delta t = 1 \text{ yr} \)

Cash at times \( t_j = j \Delta t \):

- \(-5\)
- 500
- 500
- 10,000

Discount factors \( d_{0,j} \):

\[
\begin{align*}
    d_{0,1} &= 0.9615 \\
    d_{0,2} &= 0.9070
\end{align*}
\]
3. (10 points) The textbook states the forward price formula is as follows:

Suppose an asset can be stored at zero cost and also sold short. Suppose the current spot price (at \( t = 0 \)) of the asset is \( S \). The theoretical forward price \( F \) (for delivery at \( t = T \)) is \( F = S / d(0, T) \) where \( d(0, T) \) is the discount factor between \( 0 \) and \( T \).

The proof begins: Suppose that \( F > S / d(0, T) \). Then we construct a portfolio \( \Pi_1 \) as follows: at the present time \( (t = 0) \) we borrow \( S \) amount of cash, buy one unit of the underlying asset on the spot market at price \( S \), and take a one-unit short position in the forward market.

a) The action “take a one-unit short position in the forward market” means we enter into an agreement to perform some action. When will this action take place and what will it be?

b) What is the total initial cost of this portfolio \( \Pi_1 \)?

c) What is the value of the portfolio at time \( t = T \)?

d) Explain why our ability to construct \( \Pi_1 \) violates no arbitrage.

(a) The action is to sell a forward; in other words, to agree to sell asset \( S \) at price \( F \) at time \( T \).

(b) \( GF_0 = +S_0 - S_0 + 0 = 0 \)

(c) \( \Pi_T = \frac{-S_0}{d(0, T)} + S_T - (S_T - F) \)

\[ \begin{align*}
\text{pay back borrowed cash} & \quad \text{asset} & \quad \text{forward} \\
= \frac{-S_0}{d(0, T)} + S_T - S_T + F & = F - \frac{S_0}{d(0, T)} > 0.
\end{align*} \]

(d) We know from our initial assumptions that \( F > S / d(0, T) \). Therefore, the value of \( \Pi_T \) is greater than zero, guaranteeing a positive future cash flow with zero risk and zero initial investment.
4. (5 points) (559 only)

The proof of the forward price formula continues. Suppose that \( F < S/d(0,T) \). Then we construct a portfolio \( \Pi_2 \) as follows: at the present time \((t = 0)\) we borrow the asset from someone who plans to store it during this period. We sell the borrowed asset at the spot price \( S \). We lend the proceeds \( S \) for the period from time \( 0 \) to \( T \). Long a forward contract.

a) What is the total initial cost of this portfolio \( \Pi_2 \)?

b) What actions do we take at time \( t = T \)?

c) Explain why our ability to construct \( \Pi_2 \) violates no arbitrage.

d) If the person lending the asset must be paid a fee of 0.01\( S \) at \( t = 0 \), how does this affect the no arbitrage argument?

\[ \text{Initial cost: } S - S = 0. \]

\[ \text{At time } t = T, \text{ we buy back the asset with price at } F, \text{ and return the asset to owner. We also get the pay from lending: } S/d(0,T). \]

\[ \text{At time } t = T, \text{ the total profit is: } \frac{S}{d(0,T)} - F > 0. \]

since \( F < S/d(0,T) \). So, we profit with zero cost, which is arbitrage, violating no arbitrage.

Total profit becomes:

\[ \frac{S}{d(0,T)} - F - \frac{0.01S}{d(0,T)} = \frac{0.99S}{d(0,T)} - F. \]

So, if \( \frac{0.99S}{d(0,T)} - F \leq 0 < \frac{S}{d(0,T)} - F \),

we may not be able to profit, with \( \Pi_2 \).

especially if \( \frac{0.99S}{d(0,T)} - F = 0 \), no arbitrage.