11.2 Utility Functions
Utility: Read $\S$11.1–11.3.

St. Petersburg paradox: Nicholas Bernouilli

Consider the game: Toss a fair coin, until head comes up.

- If heads on first toss, payoff = $1$: prob. = $\frac{1}{2}$.
- If heads on second toss, payoff = $2$: prob. = $\frac{1}{4}$.
- If heads on third toss, payoff = $4$: prob. = $\frac{1}{8}$.
- " fourth": $8$: prob $= \frac{1}{16}$.
- " kth": $2^{k-1}$: prob $= \frac{1}{2^k}$.

What are you willing to pay, to participate in this game?

If you think in terms of expected payoff, you would pay any amount < expected payoff.

But

\[
\text{Expected payoff} = \frac{1}{2} \times \frac{1}{2} + \frac{2}{4} \times \frac{1}{4} + \frac{4}{8} \times \frac{1}{8} + \frac{8}{16} \times \frac{1}{16} + \cdots
\]

\[
= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \cdots = \infty
\]
Daniel Bernoulli (cousin)

Value of game to player, as utility it yields.

\[ U(x) = \text{utility of wealth level } x. \]

Instead of looking at \( \sum_k p_k x_k \int \frac{1}{\text{prob. of outcome } x_k} E(X) \),

look instead at \( \sum_k p_k U(x_k) = E(U(X)) \).

D. B. suggested \( U(x) = \ln x \).

For this choice, \( \sum_{k=1}^{\infty} p_k U(x_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} \ln \left( \frac{x}{2^{k-1}} \right) \)

\[ = \left( \sum_{k=1}^{\infty} \frac{k-1}{2^k} \right) \ln 2 = (1) \ln 2. \]

But: \( U(2) = \ln 2 = \sum_{k=1}^{\infty} p_k U(x_k) = E(U(X)) \)

Suggests game is worth $2.

$2 is the "certainty equivalent" of the game.
11.2 Utility Functions
Utility example: Amount $W_0$.

Can put money in a bank at a fixed interest rate.

After 1 year, payoff = $10K$.

Consider another possible investment of the $W_0$:

a risky scheme: $X$:

$X = r.v. \text{ with payoff } = \begin{cases} \frac{1}{3} & 30K \text{ with prob. } \frac{1}{3} \\ \frac{2}{3} & 4K \text{ with prob. } \frac{2}{3} \end{cases}$

Suppose your utility function is $U(X) = \sqrt{x}$.

Which investment do you choose?

Utility of bank: $U(10K) = U(10^4) = 100$.

Let compare with expected utility of risky investment.

$E[ U(X) ] = U(30K) \frac{1}{3} + U(4K) \frac{2}{3} = 99.9$

Since $100 > 99.9$, choose bank.
Expected utility of the risky investment is \( \mathbb{E}[U(x)] = 99.9 \).

What would the investor accept as a certain payoff, instead of the risky investment?

Let \( C = \text{certain payoff} : \quad U(C) = 99.9 \).

\[
\sqrt{C} = 99.9
\]

\[
C = (99.9)^2 = 9980 = 9.98K
\]

This \( C \) is called the certainty equivalent of \( X \).
Example functions $U(x)$

$U(x) = -e^{-ax}$, $a > 0$.

$U(x) < 0$ is o.k.: what's important is comparing possible outcomes.

Basic property: $\forall x < y \Rightarrow U(x) < U(y)$

Equivalently,

$U(x) = 1 - e^{-ax}$

$U(x) = \ln x$
\[ U(x) = x^b, \quad 0 < b < 1. \]

\[ U(x) = x - bx^2, \quad b > 0 \]

and \[ 0 \leq x \leq \frac{1}{2b} \]

Properties of Utility Function:

- If \( x < y \), then \( U(x) < U(y) \). \( U \) is increasing.

Can assume \( U \) continuous.
Usually, given random variables $X$ and $Y$, payoffs from two possible investments, we want to use $U$ to compare

$E[U(X)]$ and $E[U(Y)]$;

in order to choose one of $X$ or $Y$, the investment with the larger expected utility of return payoff.

Simplest utility is $U(x) = x$.

an example of risk-neutral utility.

Each extra dollar produces the same increase in utility:

$$U(x+1) = U(x) + \left[ U(x+1) - U(x) \right] = U(x) + 1.$$  

Also:

$$U(x+1) = U(x) + U'(x)(x+1) - x = U(x) + U'(x).$$
In general, expect marginal increase in utility to decrease at higher wealth levels \( x \).

Slope \( u' \) decreases as \( x \) increases, leading to a concave function \( u(x) \).

The graph of \( u(x) \) is concave if, for every pair \( x, y \) with \( x < y \), the points on the line from \( (x, u(x)) \) to \( (y, u(y)) \) fall below the graph.
Consider an investment that has payoff
\[ X = \begin{cases} x \text{ with prob. } 1-\alpha \\ y \text{ with prob. } \alpha \end{cases}, \text{ some known } \alpha, 0 < \alpha < 1. \]

\[ E[X] = (1-\alpha)x + \alpha y. \]

Decisions to be made based on \( E[U(X)] \) (versus \( E[U(Y)] \)).

\[ E[U(X)] = (1-\alpha)U(x) + \alpha U(y). \]

Consider an alternate investment without risk, that pays off \((1-\alpha)x + \alpha y = x_\alpha.\)

To the investor, this certain payoff has utility \(U((1-\alpha)x + \alpha y)\).

Since \( U \) is concave,
\[ (1-\alpha)U(x) + \alpha U(y) < U(x_\alpha). \]

\[ E[U(X)] < U(x_\alpha); \]

\( x_\alpha = E[X] \) is the riskless payoff in amount

\( x_\alpha = E[X] \)
11.2 Utility Functions
General property: \( x < y \Rightarrow U(x) < U(y) \)

Because \( U \) is increasing and continuous, for each \( z \) in range \( U \), there is an \( x \) in domain \( U \), such that

\[
U(x) = z.
\]

This \( z = U^{-1}(z) \) defines inverse.
A game has two choices:

1. Roll a die: payoff \( X = \begin{cases} 
  0 & \text{if 1 or 2 shows, } \\
  1 & \text{if 3 or 4 shows, } \\
  27 & \text{if 5 or 6 shows, }
\end{cases} \)

2. take a fixed payoff of \( \$8 \).

Choice 1. has expected payoff \( E[X] = 60 \times \frac{1}{3} + 41 \times \frac{1}{3} + 27 \times \frac{1}{3} \)

\[ = \$9 \frac{1}{3}. \]

Suppose our "gambler" has \( U(x) = x^{\frac{1}{3}} \).

For this individual, \( U(\$8) = (8)^{\frac{1}{3}} = 2 \).

\[ E[U(X)] = U(0) \times \frac{1}{3} + U(1) \times \frac{1}{3} + U(27) \times \frac{1}{3} \]

\[ = 0 + 1.5 + 3 \times \frac{1}{3} = 4 \frac{1}{3}. \]

This person chooses 2., because the fixed payoff has greater utility.
The person would accept a smaller certain payoff. What would the person accept as a certain payoff, and consider as equivalent to rolling the die?

Define the certainty equivalent of a risky asset, as the fixed (certain) payoff that has the same utility as the expected utility of the risky asset.

Let \( C \) be certainty equivalent of 1.

\[
U(C) = E[U(X)] = \frac{4}{3}.
\]

\[
\left( \frac{4}{3} \right)^{1/3} = \frac{4}{3} \Rightarrow C = \left( \frac{4}{3} \right)^3 = \frac{64}{27} \approx 2.37.
\]

If fixed payoff is not \( C \), person will choose fixed payoff.

If fixed payoff is less than \( C \), person will choose to toss the die.
Utility functions $U$ and $V$ are equivalent if, for all random $X$, $Y$

$$E[U(X)] < E[U(Y)]$$

if and only if

$$E[V(X)] < E[V(Y)]$$

Not hard:

If $U$ is a utility function, and $V = a + bU$, constants $a$, $b$: $b > 0$.

Then: $U$ and $V$ are equivalent. (Converse true, but harder)

Consider graph(s) of $V = a + bU$.

\[\text{vertical translation by factor } a\]

\[\text{vertical scaling by factor } b\]
Can use these two degrees of freedom to arrange for \( U \) with properties,

\[
U(A) = \$A, \quad U(B) = \$B,
\]

for chosen \( A, B \).

The Arrow-Pratt coefficient is

\[
A_u = -\frac{U''(x)}{U'(x)} \quad \text{is a "measure" of risk aversion}
\]

For \( V = a + bU \), easy to show

\[
A_V = -\frac{V''(x)}{V'(x)} = -\frac{U''(x)}{U'(x)} = A_u : \quad \text{equal Arrow-Pratt coefficient}.
\]
Computing $E[U(x)]$.

If $X$ takes on discrete values: $X = \{x_i \text{ with prob } p_i : i \leq n \}$

$$E[U(x)] = \sum_{k=1}^{n} U(x_k) p_k$$

If $X$ takes on a continuous random values, let $p(x)$ be probability density for $X$:

Then $$E[X] = \int_{-\infty}^{\infty} x \cdot p(x) \, dx$$

and $$E[U(x)] = \int_{-\infty}^{\infty} U(x) \cdot p(x) \, dx.$$
11.3 Risk Aversion
Defn: Arrow-Pratt coefficient

\[ A(x) = - \frac{U''(x)}{u'(x)} \]

Larger values ⇒ more risk averse

Can (sometimes) find formula for \( U(x) \), by integrating

\[ \frac{U''(x)}{u'(x)} = -A(x); \quad \frac{d}{dx}(U') = -A(x)(U') \]
Specification of utility function.

Suppose lotto pays: $50,000 w/ prob. 1/2
 $500 w/ prob. 1/2.

Ask respondent what (s)he will pay,
in exchange for this payoff.

(s)he answers $20,000.

We now know \( \U(20,000) = \E[U(X)] \)

\[ = \U(50,000) \cdot \frac{1}{2} + \U(500) \cdot \frac{1}{2} \]

Model \( \U(x) = x^b \); measure \( x \) in \( \$K \).

\[ 20^6 = 50^{ \frac{1}{2} } + (\frac{1}{2})^{ \frac{1}{2} } \]

Numerically...

\[ \text{find} \]
\[ b = 72 \]
\[ \Rightarrow \U(x) = x^{0.72} \]
Fit $U(x)=x^b$ using a single certainty equivalent pair program, graph
Fit $U(x)=a x^{\gamma} + c$ to Sybil’s certainty equivalent pairs html, program, graph
11.4 Specification of Utility Functions
Sybil, venture capitalist, wants to make her utility for explicit consultant asks her to consider lotteries with outcomes of either $1M or $9M.

She is asked,

"If the prob. of receiving $1M is $p = 0$,
and of $9M is $q = 1$,
what is your certainty equivalent?"

Only possible answer: lottery is worth $9M.

She is asked:

"If the prob. of receiving $1M is $p = 0.1$
and of $9M is $q = 0.9$,
what is your certainty equivalent?"

She answers: $7.84 M$. 

She is asked, "If the prob. of receiving $1M is p = 0.2$
and of $9M is q = 0.8$
what is your certainty equivalent?"
She answers: $6.76M$.

Questions continue...

Finally, "If prob. of receiving $1M is p = 1$
and of $9M is q = 0$,
what is your certainty again?"
Only possible answer: $1M$. 
Find \( U(x) \) such that:

\[
U(A) = A \quad \text{and} \quad U(B) = B,
\]

\[
A = \$9 \text{M,} \quad B = \$9 \text{M.}
\]

Assume form,

\[
U(x) = ax^\gamma + c, \quad a, b, c \text{ const}.
\]

\[
U(A) = aA^\gamma + c = A \quad \Rightarrow \quad \begin{bmatrix} A^\gamma & 1 \end{bmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} A \end{pmatrix}
\]

\[
U(B) = aB^\gamma + c = B
\]

Cramer's rule:

\[
a = \frac{\det \begin{bmatrix} A & 1 \\ B & 1 \end{bmatrix}}{\det \begin{bmatrix} A^\gamma & 1 \\ B^\gamma & 1 \end{bmatrix}}
\]

\[
= \frac{(A-B)}{(A^\gamma - B^\gamma)} = \frac{(B-A)}{(B^\gamma - A^\gamma)}
\]

\[
c = \frac{\det \begin{bmatrix} A^\gamma & A \\ B^\gamma & B \end{bmatrix}}{\det \begin{bmatrix} A^\gamma & 1 \\ B^\gamma & 1 \end{bmatrix}}
\]

\[
= \frac{[A^\gamma B - B^\gamma A]}{(A^\gamma - B^\gamma)}.
\]
Link visited in class

Fit $U(x)=x^b$ using a single certainty equivalent pair program, graph
Fit $U(x)=a x^\gamma + c$ to Sybil’s certainty equivalent pairs html, program, graph
11.5 Utility Functions and the Mean-Variance Criterion
Combine Utility with Markowitz problem

(No risk free; 3 risky assets)

Return rates \( r_1, r_2, r_3 \)

Random vars

\[ E[r_1] = \tilde{\mu}_1, \quad E[r_2] = \tilde{\mu}_2, \quad E[r_3] = \tilde{\mu}_3, \]

\[ \text{var}(r_i) = \sigma_i^2, \quad \text{var}(r_2) = \sigma_2^2, \quad \text{etc}. \]

\[ \text{cov}(r_i, r_j) = \rho_{ij} : \text{given}. \]

Suppose portfolio with initial value \( W_0 \).

After \( t \) yr, \( y = (1+r)W_0 \) where \( r \) is return rate for portfolio.

Portfolio weights \( w_1, w_2, w_3 \) such that \( w_1 + w_2 + w_3 = 1 \)

and \( w_1 \geq 0, w_2 \geq 0, w_3 \geq 0. \)

(No shorting)
Consider utility $U(y)$: $y = (1 + \gamma)w_0$

$= (1 + w_1 \bar{\gamma}_1 + w_2 \bar{\gamma}_2 + w_3 \bar{\gamma}_3)w_0$.

One problem to consider is:

maximize $E[U(y)]$,

with constraints

$w_1 + w_2 + w_3 = 1$,

$E[y] = \bar{y}$, where $\bar{y}$ is given *(in book, M)*.

But: $E[y] = E[(1 + \gamma)w_0] = (1 + \bar{\gamma})w_0$ where

$\bar{\gamma} = E[\gamma]$

$= E[w_1 \gamma_1 + w_2 \gamma_2 + w_3 \gamma_3]$

$= w_1 \bar{\gamma}_1 + w_2 \bar{\gamma}_2 + w_3 \bar{\gamma}_3$.

The constraint $E[y] = \bar{y}$

is equivalent to: $\bar{y} \leq w_1 \bar{\gamma}_1 + w_2 \bar{\gamma}_2 + w_3 \bar{\gamma}_3 = \bar{\gamma}$, where $\bar{\gamma} \bar{\gamma} = \bar{y} / w_0$,

$\bar{\gamma} = \frac{\bar{y}}{w_0} - 1$. 
Special case: \( U(y) = ay - \frac{b}{2} y^2 \)

restrict to \( y \) where \( U'(y) > 0 \)

\[
E[U(y)] = a E[y] - \frac{b}{2} E[y^2],
\]

\( y \) is r.v.

For r.v. \( y \), \( \text{var}(y) = E[y^2] - \bar{y}^2 \)

\( (\text{var}(y) = E[(y-\bar{y})^2]) \)

\[
E[U(y)] = a \bar{y} - \frac{b}{2} (\bar{y}^2 + \text{var}(y)).
\]

and constraints, \( w_1 + w_2 + w_3 = 1 \),

\[
w_1 \bar{x}_1 + w_2 \bar{x}_2 + w_3 \bar{x}_3 = \bar{\bar{x}} = \left( \frac{\bar{y}}{w_0} - 1 \right)
\]

So: \( \bar{y} \) is being held constant; \( ay - \frac{b}{2} \bar{y}^2 \) is held constant

max. \( E[U(y)] \) is obtained by minimizing \( \text{var}(y) \)

know how to do!
Another special case:

If returns are normal random vars.
then (claim) Markowitz approach (min var)
and utility function approach
both give same sol'n, for any risk-averse util. fn.

Let $U$ be a util. fn:
suppose $y$ is normal, mean $M$, $\text{var}(y) = b^2$.
Notice $M, b^2$ completely determine the prob. distribution.
then $M, b^2$ completely determine $E[U(y)]$:

$$E[U(y)] = f(M, b).$$

$f$ is an increasing fn of $M$, a decreasing fn of $b$.

Portfolio is a linear comb. of normal r. v.'s,
hence also normal r. v.; so get an $f$ for portfolio

For fixed $\bar{y} = M$, maximize $f(M, b)$ by minimizing $b = b(w_1, w_2, w_3)$.

Take min $b(w, w_2, w_3)$, subject to constraints.
Links visited in class

Feasible set for Markowitz problem sigma_rbar_three_asset_noshorting. mp4, html
11.6 Linear Pricing
11.6 Arbitrage Example (Actual event, Spring 2016)

$CDN \leftrightarrow US exchange rate in bank is: $1 CDN = 0.70 US.

NYS Thruway: considers $5 CDN (bill) = 4 US.

Motorist:
    travels WInstville \rightarrow Depew; gives $5 CDN to collector
    at Depew.

    Collector: returns 4 US - 0.15 US
    = 3.85 US.

    Motorist: takes 3.85 US to a bank
    and exchanges for \( \frac{3.85}{0.70} = 5.50 \) CDN.

Immediate profit: 0.50 CDN.

This is an investment that gives immediate positive reward, with no future obligations. Type A arbitrage.
11.6 "BOGO" arbitrage example:

Shoe store has a "Buy one, get one for 50% off" sale.

Investor finds two people who each want 1 pair of shoes.

Investor borrows \((P + 0.5P)\) from bank; \(P =\) price of a pair of shoes

- pays 1.5\(P\) for two pairs.
- sells each pair, for \(P\):
  - now has 2\(P\) in hand.
- repays bank 1.5\(P\)

Immediate profit: \(2P - 1.5P = 0.5P\).

Also: type A arbitrage,

an investment that has no (or negative) initial cost,
but that has positive probability of positive payoff in future, is called type B arbitrage.
Linear pricing

Let $d$ be a security with price $P$.
Let $2d$ be the security that always pays twice what $d$ pays.
Let $Q = \text{price of } 2d$.

If $Q > 2P$
- borrow $2P$ from bank
- use $2P$ to buy two copies of $d$
- bundle two copies into $2d$
- sell $2d$ for $Q$
  now have $Q$ in hand.
- pay back bank with $2P$
- immediate profit $(Q-2P) > 0$

If $Q < 2P$
- borrow $Q$
- purchase $2d$ with $Q$.
- break up (unbundle)
  $2d$ into two copies of $d$
- sell each copy for $P$
  now have $2P$ in hand.
- pay back bank with $Q$
- immediate profit $(2P-Q) > 0$.

If assume no arbitrage is possible,
must be: both $Q > 2P$ false, and $Q < 2P$ false.

$Q \leq 2P$ and $Q \geq 2P \Rightarrow Q = 2P$. 
For securities that can be divided into fractions, a similar argument:

If \( d_1 \) = security with price \( P_1 \),
\( d_2 \) = security with price \( P_2 \),
then price of \( \alpha d_1 + \beta d_2 \) must be \( \alpha P_1 + \beta P_2 \).

(Linear Pricing)
12.2 Forward Contracts
12.1 Forward Contracts

Forward example: Hops.

Suppose current price of hops is $15/lb.
A brewer will need hops to make beer, in 6 mos.

If price of hops jumps over next 6 mos —
- the brewer can raise the price of beer
- take a loss in profit.

Brewer: finds a seller, and agrees to buy
1000 lbs of hops for $16/lb in 6 mos.

Now: brewer is protected against a big rise in price of hops.
Good decision if price goes up.
Seller can make money if price goes down,
or stay the same.

Buyer is long 1000 lbs, seller is short 1000 lbs: spot price vs. delivery price for immediate
12.3 Forward Prices
P.V. of $F_t$ is $F_0$.

Let $e_p$ be the price of the bond at time $t$.

Assume bond is delivered at $t=M$.

Let $F_t = \text{forward price of bond, to be delivered at } t=M$.

Let $F_t$ be the price of the bond at time $t$.

If the bond is bought at time $t$, its delivery price at time $t+M$ is $F_t$.

If the bond is sold at time $t$, it pays off at time $M$.

The bond pays off at time $M$.

The bond is delivered at time $M$.

The bond is purchased at time $t$.

The bond is sold at time $t$.

The bond is bought at time $t$.

The bond is sold at time $t$.

The bond is bought at time $t$.

The bond is sold at time $t$.

The bond is bought at time $t$.
Forward price \( F_m = \frac{1}{d_{om}} \left( S - c_1 d_{o1} - c_2 d_{o2} - \cdots - c_m d_{om} \right) \).

Can simplify if identities \( d_{ij} = d_{ik} d_{kj} \) available.

**Example:** Here, \( c_1, c_2, \ldots, c_m \) are positive asset with carrying costs.

**Gold:** Suppose \( S = \$1236/ounce \).

Suppose storage cost is \$8 per ounce per year:

payable quarterly in advance.

Suppose interest rate is \( r = 2\% \), compounded quarterly.

What is the forward price of gold, for delivery in 1 year?
\[ \Delta t = 0.25 \text{ year} \]

\[ \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \uparrow \\
\text{Cost of storage} & \$ -2 & -2 & -2 & -2 & + F_4 \\
\end{array} \]

Income from sale of gold in future:

\[ P.V. = P.V. \text{ of income stream} \]

\[ = -2 \cdot d_{01} -2 d_{02} -2 d_{03} + F_4 d_{04} \]

Spot price is \( S = \$1236 \).

Since \( S = P.V. \text{ of income stream} \),

\[ S = -2(1 + d_{01} + d_{02} + d_{03}) + F_4 d_{04} \]

\[ \Rightarrow F_4 = S + 2 \left( 1 + d_{01} + d_{02} + d_{03} \right) \]

\[ d_{04} \]

More generally:

\[ S = - \sum_{k=0}^{M-1} c_k d_{0k} + F_M d_{0M} \quad ; \quad F_M = \frac{1}{d_{0M}} \left( S + \sum_{k=0}^{M-1} c_k d_{0k} \right) \]
<table>
<thead>
<tr>
<th>Delta t</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25000</td>
<td>0.02000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>time t</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d00</th>
<th>d01</th>
<th>d02</th>
<th>d03</th>
<th>d04</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>0.99502</td>
<td>0.99007</td>
<td>0.98515</td>
<td>0.98025</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c0</th>
<th>c1</th>
<th>c2</th>
<th>c3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.00000</td>
<td>-2.00000</td>
<td>-2.00000</td>
<td>-2.00000</td>
</tr>
</tbody>
</table>

\[
\sum_{k=0}^{3} c_k d_{0k} \\
\frac{\sum_{k=0}^{3} c_k d_{0k}}{d_{04}} \\
\frac{S}{d_{04}} \\
S - \sum_{k=0}^{3} c_k d_{0k} \\
F_4 \\
F_4 d_{04}
\]
12.3 Forward Prices
Assumptions:

a) no transaction costs
b) assets can be divided arbitrarily
c) no storage costs (for now)
d) can sell asset short
Claim: $F = \frac{S}{d(0,T)} \Leftarrow \text{spot price}
\Rightarrow \text{discount factor}: d(0,T) < 1$

\( \text{If } F > \frac{S}{d(0,T)} \)

- "long" the asset: borrow $S$ from bank
- buy asset for $S$
- arrange a forward contract to sell asset for $F$ at $t = T$.

at $t = 0$: no initial outlay

at $t = T$: exercise the forward contract: deliver the asset and receive $F$.
- repay loan: pay $\frac{S}{d(0,T)}$.

profit at $t = T$ is

$$F - \frac{S}{d(0,T)} > 0: \text{arbitrage (type B)}$$

\( \therefore \text{if no arbitrage, must have } F - \frac{S}{d(0,T)} \leq 0. \)

\( \text{If } F < \frac{S}{d(0,T)} \)

- "Short" the asset: borrow the asset from someone who owns it but plans to store it.
- sell the asset for $S$.
- put funds $S$ in bank.
- arrange a forward contract to buy asset for $F$ at $t = T$.

at $t = T$: withdraw $\frac{S}{d(0,T)}$ from bank.

exercise the forward contract to buy the asset for $F$;
return asset to owner.

profit at $t = T$ is

$$\frac{S}{d(0,T)} - F > 0: \text{arbitrage.}$$

\( \therefore \text{if no arb., } F \geq \frac{S}{d(0,T)}. \)
Discount factors

notation \( d(a, b) \) converts cash at time \( b \),

to cash at time \( a \):

\[
(cash \ at \ time \ a) = (cash \ at \ time \ b) \times d(a, b).
\]

normally, \( a < b \), \( d(a, b) < 1 \)

notation \( d_{ij} = d(i_\Delta t, j_\Delta t) \)

\( \Delta t = \frac{1}{2} \text{ year} \):

\[
d_{01} = d(0, \frac{\Delta t}{2}) : \text{ converts cash at } t = \frac{1}{2} \text{ year}
\]

to cash at \( t = 0 \)

\[
d_{02} = d(0, 2\Delta t)
\]

identity:

\[
d_{ij} = d_{ik} \times d_{kj}
\]

\( k = i \Delta t \), \( k = j \Delta t \)

\( x \times d_{ik} \)

\( x \times d_{kj} \)

\( x \times d_{ij} \)
Consider a 5-year bond, $1000 face value, pays $70 coupon once per year.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>now</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$70</td>
<td>$70</td>
<td>$70</td>
<td>$70</td>
<td>$70</td>
</tr>
</tbody>
</table>

Present value using discount factors:

$$P.V. = 70d_0 + 70d_1 + 70d_2 + 70d_3 + 70d_4 + 1000d_5$$

(If $S$ = spot price of bond,
$S = P.V.$)
Consider this same bond, but purchased 2 years from now.

\[ \text{now} = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ 70 \quad 70 \quad 70 \]

\[ 1000 \]

\[ F_2 = \text{forward price for delivery in 2 years, of bond.} \]

\[ = \text{value 2 years from now, of income stream shown} \]

\[ = 70 \ d_{23} + 70 \ d_{24} + 70 \ d_{25} + 1000 \ d_{25} \]

Recall: \( d_{ij} = d_{ji}, \ i \leq i \leq j \).

Mult. by \( d_{02} \)

\[ F_2 \ d_{02} = 70 \ d_{02} \ d_{23} + 70 \ d_{02} \ d_{24} + (70 + 1000) \ d_{02} \ d_{25} \]

\[ = 70 \ d_{03} + 70 \ d_{04} + (70 + 1000) \ d_{05} \]
But: \[ S = 70d_{o1} + 70d_{o2} + 70d_{o3} + 70d_{o4} + (70 + 1000)d_{o5} \]

\[ F_2d_{o2} \]

\[ S - F_2d_{o2} = 70d_{o1} + 70d_{o2}. \]

\[ F_2 = \frac{1}{d_{o2}} (S - 70d_{o1} - 70d_{o2}) \]

\[ = \frac{S}{d_{o2}} - 70 \frac{d_{o1}}{d_{o2}} - 70 \frac{d_{o2}}{d_{o2}} \]

\[ d_{o2} = d_{o1}d_{i2} \]

\[ \therefore \frac{d_{o1}}{d_{o2}} = \frac{1}{d_{i2}}. \]

\[ F_2 = \frac{S}{d_{o2}} - 70 \cdot \frac{1}{d_{i2}} - 70. \]
12.4 The Value of a Forward Contract
Holding a forward contract, may have positive or negative value. At $t = 0$, a price $F(0)$ is written into contract. At $t = T$, the buyer will pay $F(0)$ to seller. The contract is signed at $t = 0$ with no exchange of funds. As time goes on, the spot price for the same asset changes with time, but $F(0)$ in contract is fixed. Holding the contract at time $t = T$, is worth $f(T) = S(T) - F(0)$ to the buyer. At $t = 0$, the value of the contract is $f(0) = 0$.

Claim: $f(t) = (F(t) - F(0)) dt (t, T)$, the value of contract at time $t$. Price written into contracts for same asset, delivery but at time $t$. 

value to buyer of \( f(t) \) asset, of holding contract.

This \( f(t) \), is an example of a derivative.
12.5 Swaps
12.5 Swaps

A swap is an agreement to exchange future cash flow streams.

"Plain Vanilla" interest rate swap.

Two parties, usually banks, agree to swap interest payments.

For example: fixed rate payments, for adjustable rate payments.

Banks agree on notional principal: (also, notional)

Notional principal not exchanged, but will serve as figure to base payments.

One party pays fixed rate payments based on notional payments.

Other party pays adjustable rates based on some benchmark like London Interbank Offered Rate (LIBOR) rate, wholesale by bank or bank...
Bank A pays B
fixed rate \( r \)
payments, based
on notional \( N \)
B receives $

P.V. of stream

\[
= rN d_0 + rN d_0 + rN d_3 + rN d_4
\]

\[
= rN \sum_{j=1}^{4} d_{0j}
\]

Bank B pays A
floating rate \( c_i \)
payments, based
on notional \( N \)
c is known at \( t=0 \):
rate for lending
from \( t=0 \) to \( t=1 \).
c is known at \( t=j \)
rate for lending from \( t=j \) to \( t=j+1 \)
12.5 Swaps
Don’t know: r. What is fair value of a?

Due to not knowing R, c, \( c_1, c_2, c_3 \)

\( c_1 \) = actual interest rate at \( t = 1 \),

for funds lent at \( t = 1 \), for period of 1 yr.

Do know: discount factors \( d_{01}, d_{02}, d_{03} \)

from FTLT-bill auctions.

From \( d_{01}, d_{02}, d_{03}, \) can

estimate "short rates" \( r_0, r_1, r_2, r_3 \)

the interest rates for funds lent at

\( r_0 \): time 0, for 1 yr

\( r_1 \): time 1, for 1 yr

\( r_2 \): time 2, for 1 yr.
Given $d_{01}$, find $n_0$

\[(100d_{01})(1+n_0) = 100 \quad \Rightarrow \quad 1 + n_0 = \frac{1}{d_{01}} \quad \text{and} \quad n_0 = \frac{1}{d_{01}} - 1\]

If $d_{01} = .9$, then \[n_0 = \frac{1}{.9} - 1 = \frac{10}{9} - 1 = \frac{1}{9}\]

Given $d_{01}$, $d_{02}$ find $n_1$

\[(100d_{12})(1+n_1) = 100 \quad \Rightarrow \quad d_{12} = \frac{1}{1+n_1} \quad \text{and} \quad n_1 = \frac{1}{d_{12}} - 1\]

Suppose $d_{01} = .9$, $d_{02} = .72$. Then $d_{01}d_{12} = d_{02}$, so $d_{12} = \frac{d_{02}}{d_{01}} = \frac{.72}{.9} = .8$

\[\frac{1}{d_{12}} = \frac{1}{.8} = \frac{5}{4} \quad \text{and} \quad n_1 = \frac{5}{4} - 1 = \frac{1}{4}\]
\[ B \text{ pays } A \]

\[ \text{estimated } \begin{array}{cccc}
\tau_0 N & \tau_1 N & \tau_2 N & \tau_3 N \\
\text{ (actual } c_0 N \text{) } & \text{ (actual } c_1 N \text{) } & \text{ (actual } c_2 N \text{) }
\end{array} \]

\[ \text{P.V. of estimated stream} \]

\[ = (\tau_0 N) d_{01} + (\tau_1 N) d_{02} + (\tau_2 N) d_{03} + (\tau_3 N) d_{04} \]

\[ = N (\tau_0 d_{01} + \tau_1 d_{02} + \tau_2 d_{03} + \tau_3 d_{04}) \]

\[ = N \sum_{i=0}^{3} \tau_i \, d_{0,i+1} \quad \Leftarrow = N (1 - d_{04}) \]

where:

\[ d_{01} = \frac{1}{1 + \tau_0} \quad d_{02} = \frac{1}{(1 + \tau_0)(1 + \tau_1)} = d_{01} \cdot d_{12} \]

\[ d_{03} = \frac{1}{(1 + \tau_0)(1 + \tau_1)(1 + \tau_2)} = d_{01} \cdot d_{12} \cdot d_{13} \]

\[ d_{04} = \frac{1}{(1 + \tau_0)(1 + \tau_1)(1 + \tau_2)(1 + \tau_3)} \]

Lemma: If \[ d_{0,i+1} = \frac{1}{(1 + \tau_0)(1 + \tau_1) \cdots (1 + \tau_i)} \] then \[ \sum_{i=0}^{M-1} \tau_i \, d_{0,i+1} = 1 - d_{0M} \]

for \( M = 1, 2, \ldots \).
Recall: A pays B $r \mathcal{N}\left( \sum_{i=1}^{4} d_{0i} \right)$.

Choose $r$ such that

$$r \mathcal{N}\left( \sum_{i=1}^{4} d_{0i} \right) = \mathcal{N}(1 - d_{04}).$$
12.5 Swaps
Commodity Swap:

Party A agrees to buy from party B,

$U$ units of a commodity, at fixed price $X$ per unit,

for $M$ months, starting in 1 month from now.

\[ \Delta t = \frac{1}{12} \text{yr} \]

0 \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 3 \hspace{1cm} M \text{ months}.

A's cash flow:

\[ -UX \quad -UX \quad \ldots \quad -UX \]

stream pays

A sells on (commodity) + US$_1$ + US$_2$ + ...

spot market

where S$_1$, S$_2$, ..., S$_M$ are prices on spot market

for 1 unit of commodity.

Don't know S$_1$, S$_2$, ..., S$_M$.

Do know F$_1$, F$_2$, ..., F$_M$ : F$_i$ = price written into a

forward contract now (t=0)

for delivery of 1 unit of U, at t=0
To estimate P.V. of cash stream to party A:

P.V. of swap to A, is:

\[ \sum_{i=1}^{M} \left( \frac{U F_i - UX}{d_{0,i}} \right) \]

\[ \text{estimated discount factor} \]

\[ \text{cash in - cash out to change cash} \]

\[ \text{at } t = t_i \]

\[ \text{at } t_i \text{ to cash now (t=c)} \]

When writing contract for commodity swap,

choose \( X \) so that

\[ X \left( \sum_{i=1}^{M} d_{0,i} \right) = \sum_{i=1}^{M} F_i \cdot d_{0,i} \]

\[ \text{this is the fair price X for the commodity.} \]
12.6 Basics of Futures Contracts
Futures are similar in goals to Forwards.

Purpose of futures market:
- bring buyers & sellers together,
- deal with book-keeping and price fluctuations.

At mid-July delivery of corn: 1 unit = 5000 bushels.

Suppose I want 1 unit:
If using a Forward contract, my broker calls a farmer willing to sell.
They agree on a price $F_0$.

Tomorrow, someone else calls a broker who calls the farmer
and they agree on a price $F_1$.

Etc.
Farmers have lots of orders, lots of different prices.
I also have lots of different prices, if orders on different dates.

Futures market accounts for such fluctuations, & pays immediately.
12.6 Basics of Futures Contracts
Futures market: accounts for price fluctuations, 
so pays immediately.

Everyone who trades, has a margin account.

Every day, if a price change leads to a gain, 
the gain is added to the margin account.

If a price change leads to a loss, 
the loss is taken off the margin account.

If price today is $5.23/1 unit of commodity 
and I hold 1000 units; 
and tomorrow price changes to $5.25/1 unit, 
then I get 
$\left(5.25 - 5.23\right) \times 1000 = 20 
added to my margin account.
maintenance level:

If margin account goes below this level:

must either 1. close out position & accept the loss
         2. replenish margin account to required maintenance level.

If account rises above maintenance level, can take out extra.

If there is a delivery (the day in mid-July arrives),
the futures price is the price on the delivery day,
and gain/loss is (already) in margin account.

Process of adjusting margin accounts is called "marking to market."
Let $F$ be forwards prices, $F$ be futures prices.
Consider two strategies, B and A.

\[ t = 0 \quad t = \Delta t \quad t = m \Delta t = T \]

**Strategy B**

At $t = 0$, arrange a forward contract; agree to purchase 1 unit of asset at $T = m \Delta t$, for $F_0$.

At $t = T$, profit = $\Delta S(T) - F_0$

↑

Spot price of commodity/asset at $T$. 
Strategy A

\[ T = M \Delta t \]

At \( t = 0 \), go long \( d_{1M} \) units of future, for delivery at \( T = M \Delta t \) (purchase).

At \( t = t_1 = \Delta t \), price changes from \( F_0 \) to \( F_1 \).

Margin account is changed by \( (F_1 - F_0) d_{1M} \) and go long \( (d_{2M} - d_{1M}) \) units of future, same delivery \( T \).

Now hold \( d_{1M} + (d_{2M} - d_{1M}) = d_{2M} \) units of future.

\[ \text{If } F_1 > F_0, \text{ take } (F_1 - F_0) d_{1M} \text{ from margin account to invest in bank.} \]

\[ \text{If } F_1 < F_0, \text{ borrow } (F_0 - F_1) d_{1M} \text{ from bank to add to margin account.} \]
Strategy A (std).

at \( t = t_2 = 24t \): price changes from \( F_i \) to \( F_2 \).

- Margin account is charged by \( \$ (F_2 - F_i) d_{2M} \).

- Go long \( (d_{3M} - d_{2M}) \) units of future, same delivery.

- Now hold \( d_{2M} + (d_{3M} - d_{2M}) = d_{3M} \) units of future.

Continue until \( t = t_{M-1} = (M-1)\Delta t \).

\[
\begin{array}{cccc}
\Delta t & 1 & \cdots & T \\
0 & (F_i - F_0) d_{2M} & (F_{M} - F_{M-1}) d_{M-1, M} & (F_{M} - F_{M-1}) d_{M-1, M} \\
\text{Cash to Bank} & (F_{M} - F_{M-1}) d_{M-1, M} & \text{Cash to Bank} & (F_{M} - F_{M-1}) d_{M-1, M} \\
\end{array}
\]

Value of cash to bank at \( t = M \Delta t \):

\[
\begin{align*}
\text{is:} & \quad (F_i - F_0) d_{2M} \cdot \frac{1}{2} + \cdots + (F_{M} - F_{M-1}) d_{M-1, M} \cdot \frac{1}{2} + \cdots + (F_{M} - F_{M-1}) \\
& \quad = -F_0 + 0 + c + \cdots + F_M \quad \text{But} \quad F_M = S(T) \\
\end{align*}
\]

\[c. \quad S(T) - F_0.\]
If \( F_0 \neq F_0 \), there is an arbitrage opportunity by combining strategies A & B.

Theoretically \( F_0 = F_0 \).
12.6 Basics of Futures Contracts
Actual relationship between futures \( F_0 \) and forwards \( F^* \):

- If interest rates "deterministic", \( F_0 = F^* \).

- If interest rates change, and spot prices & interest rates are positively correlated, \( F_0 > F^* \).

- If spot prices & interest rates change, and are negatively correlated, \( F_0 < F^* \).

Futures are convenient, if asset is available on a market.
12.8 Relation to Expected Spot Price
Relationship between futures $F$, and spot prices $S(T)$.

$F = F_0$ = price for purchase of 1 unit of asset at time $T$.

agreed price now, $t = 0$

$S(T)$ = spot price at time $T$: unknown, treat as random variable.

$E[S(T)] = \text{expected value}$

Expect $F_0 = E[S(T)]$ would hold.

If: $F > E[S(T)]$, called contango

If: $F < E[S(T)]$, called \underline{normal} backwardation.

Two types of investors:

speculators & hedgers.
12.9 The Perfect Hedge
Speculators: not really interested in delivery of asset (corn/oil,...) instead, want to make money on fluctuations in prices; will close out position before T.

Hedgers: actually want to buy or sell asset (corn/oil,...) and are looking for price certainty, are willing to take "losing" positions. will go long even if \( F > E[S(T)] \).
(ex/ baker needs corn for muffins in 6 mos.)

or will go short, even if \( F < E[S(T)] \).
(ex/ corn supplier needs funds in 6 mos., for equipment payment)

Speculators: will not go long if \( F > E[S(T)] \), not go short if \( F < E[S(T)] \).
Currency hedge.
Suppose I will need €5000 in December.

Present exchange rate: \( 1 \, \text{€} = 1.11 \, \text{US} \). (2016-02-19).

Futures market: at present, for delivery of \( 1 \, \text{€} \) in Dec.,
rate is \( 1 \, \text{€} = 1.12 \, \text{US} \).

Can hedge against € increasing
by purchasing now at 1.11 US; & storing until needed.

Can hedge, by purchasing futures now, for delivery in Dec.

Not all currencies have futures;
however there is a strong correlation between currencies.
Hedging:

ex: I make corn bread. Someone wants delivery in late Dec. I will need 5000 bu. of corn, but can't store it.

To figure price for corn bread, need to know how much the corn will cost.

Could arrange a forward contract, but easier to arrange a futures contract, at $6.20/bu or: $(5000)(6.20) = 31,000.$

Now know with certainty, amount to be paid for corn.
Perfect hedge:
An "ideal" financial position, in which
an investor eliminates uncertainty
Usually, not possible to construct "perfect hedge."

1/ small quantities of corn, or non-integer multiples
   of (5000) bu.

1/ may not find futures contract for our delivery date.

1/ delivery might not be in right location.

1/ commodity might not be on a futures market.

Basis risk: what remains of position
still "random", after hedging.
12.10 The Minimum-Variance Hedge
Minimum Variance Hedge:

Let \( x \) = cash flow at time \( T \).

exp will purchase \( W \) units of a commodity at spot price \( S(T) \), at time \( T \).

Then: \( x = -WS(T) \).  (reg. for expenditure).

To hedge these expenditures, use \( h \) units of a future.

\[
F(0) = \text{price/unit (now)} \quad F(T) = \text{price/unit (at } T)\]

at \( T \):

\[
y = x + (F(T) - F(0))h
\]

\( y \) gain or loss from \( h \) units of future.
12.10 The Minimum-Variance Hedge
Recall 12.4

Value at time $t$ of holding a forwards contract is

$$V(t) = (F(t) - F(0)) d(t, T)$$

A contract arranged at $t=0$ has value

$$(F(T) - F(0)) \text{ at delivery}.$$ 

Now: use futures for hedging.

treat futures like forwards.
Last day -

Planning to spend cash at time $T$
to purchase $W$ units of a commodity at $S(T)$, spot price.

$x = -W S(T)$.

To hedge: go long $h$ units of a future.

$F(0) =$ price now, $F(T) =$ price at time $T$, written into contract

Gain or loss from future will be

$(F(T) - F(0)) h$.

Net at delivery:

$y = x + (F(T) - F(0)) h$

↑
what will be spent on $W$
units of $S(T)$

↑
 gain or loss from $h$ units
of future

random: $x, F(T)$
$S(T), y$

if available, $F(T) = S(T)$?
Identities: \( X, Y \) random; \( a, b, c \) const

\[
\text{var}(x+a) = \text{var}(X) \quad \text{var}(x+a) = E[(x+\bar{x} - \bar{x}+\bar{y})^2] \\
= E[(x-\bar{x})^2] = \text{var}(X) \\
\text{var}(bX) = b^2\text{var}(X). \\
\text{var}(X+Y) = E[(X+Y - \bar{X+Y})^2] \\
= E[(X-\bar{X})+(Y-\bar{Y})]^2] \\
= E[(X-\bar{X})^2] + 2E[(X-\bar{X})(Y-\bar{Y})] + E[(Y-\bar{Y})^2] \\
= \text{var}(X) + 2\text{cov}(X, Y) + \text{var}(Y). \\
\text{cov}(cX, Y) = \text{cov}(X, cY) = c\text{cov}(X, Y). \\
\text{cov}(x+a, y+b) = \text{cov}(X, Y)
Model hedging grapefruit juice with orange juice futures.

Suppose
\[ S = a_0 + a_1 z_1 + a_2 z_2 \]
\[ F = b_0 + b_1 z_1, \]
where \( a_0, a_1, a_2 \) are constants, \( b_0, b_1 \) are constants, \( z_1, z_2 \) are random variables:
\[ E[z_1] = \bar{z}_1 = 0, \ E[z_2] = \bar{z}_2 = 0, \]
\[ \text{var}(z_1) = 1, \ \text{var}(z_2) = 1, \]
\[ \text{cov}(z_1, z_2) = 0. \]

Explanation:
Growing seasons differ:
\[ z_1 \]
\[ \text{oranges} \]
\[ \overset{\text{grapefruit}}{z_2} \]

\( S = \) price of grapefruit juice at \( T \)
\( F = \) price of orange juice at \( T. \)
Consider \( y(k) = -WS(t) + h(F(t) - F(0)) \) \( \text{const.} \)

\[
\text{var}(y(k)) = \text{var}( -WS(t) + hF(t) - hF(0))
\]

\[
= \text{var}( -WS(t) + hF(t))
\]

\[
= \text{var}( -WS(t)) + 2\text{cov}( -WS(t), hF(t)) + \text{var}(hF(t))
\]

\[
= W^2 \text{var}(S(t)) - 2Wh \text{cov}(S(t), F(t)) + h^2 \text{var}(F(t))
\]

Minimize: set \( \frac{\partial}{\partial h} \text{var}(y(k)) = 0. \)

\[
\frac{\partial}{\partial h} \text{var}(y(k)) = 0 - 2W \text{cov}(S(t), F(t)) + 2h \text{var}(F(t)).
\]

\[
\Rightarrow h = h_{\text{min}} = \frac{W \text{cov}(S(t), F(t))}{\text{var}(F(t))}
\]

Claim:

\[
\text{var}(y(h_{\text{min}})) = W^2 \left[ \frac{\text{var}(S(t)) - \left( \frac{\text{cov}(S(t), F(t))}{\text{var}(F(t))} \right)^2}{\text{var}(F(t))} \right]
\]
\[ \text{var } F = \text{var } (b_0 + b_1 \beta_1) = \text{var } (b_1 \beta_1) = b_1^2 \text{var } (\beta_1) = b_1^2. \]

\[
\text{cov } (S, F) = \text{cov } (a_1 + a_2 \beta_2, b_0 + b_1 \beta_1)
\]
\[= \text{cov } (a_1 \beta_2 + a_2 \beta_2, b_0 \beta_1 + b_1 \beta_1) \]
\[= a_1 b_1 \text{cov } (\beta_2, \beta_1) + a_2 b_1 \text{cov } (\beta_2, \beta_1) \]
\[= a_1 b_1 \text{var } (\beta_1) + a_2 b_1 \text{cov } (\beta_2, \beta_1) \]
\[= a_1 b_1 \text{var } (\beta_1) + a_2 b_1 \text{cov } (\beta_2, \beta_1) \]
\[= a_1 b_1 \text{var } (\beta_1) + a_2 b_1 \text{cov } (\beta_2, \beta_1) \]
\[= a_1 b_1 \text{var } (\beta_1) + a_2 b_1 \text{ cov } (\beta_2, \beta_1) \]
\[= 0 \text{ if weather for two periods, unrelated.} \]

\[
\text{var } (y(\text{min})) = \frac{W^2 \left[ \text{var } (S(t)) - \frac{\text{cov } (S(t), F(t))^2}{\text{var } (F(t))} \right]}{w^2} \]
\[= \frac{W^2 \left[ a_1^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right]}{w^2} = W^2 a_2^2. \]

\[y(\text{min}) \text{ remains a random variable if } a_2 \neq 0. \]
\[\text{basis risk if } a_2 \neq 0.\]