12.5 Swaps
Don't know: \( r \). What is fair value of \( r \)?

Due to not knowing \( c_0, c_1, c_2, c_3 \):

\( c_1 \) = actual interest rate at \( t = 1 \),
for funds lent at \( t = 1 \), for period of 1 yr.

Do know: discount factors \( d_0, d_1, d_2, d_3 \)

from T-bill auctions.

From \( d_0, d_1, d_2, d_3 \), can estimate "short rates" \( r_0, r_1, r_2, r_3 \)

the interest rates for funds lent at

\( r_0 \): time 0, for 1 yr
\( r_1 \): time 1, for 1 yr
\( r_2 \): time 2, for 1 yr.
Given $d_{01}$, find $n_0$

\[(100d_{01})(1 + n_0) = 100 \Rightarrow 1 + n_0 = \frac{1}{d_{01}} \text{ and } n_0 = \frac{1}{d_{01}} - 1\]

If $d_{01} = 0.9$, then $n_0 = \frac{1}{0.9} - 1 = \frac{10}{9} - 1 = \frac{1}{9}$.

Given $d_{01}$, $d_{02}$

find $n_1$

\[(100d_{12})(1 + n_1) = 100 \Rightarrow d_{12} = \frac{1}{1 + n_1}, \quad n_1 = \frac{1}{d_{12}} - 1\]

Suppose $d_{01} = 0.9$, $d_{02} = 0.72$. Then $d_{01}d_{12} = d_{02}$, so $d_{12} = \frac{d_{02}}{d_{01}} = \frac{0.72}{0.9} = 0.8$

\[\frac{1}{d_{12}} = \frac{1}{0.8} = \frac{5}{4} \text{ and } n_1 = \frac{5}{4} - 1 = \frac{1}{4}\]
B pays A

\[ \text{estimated } \begin{array}{cccc}
\tau_0 N & \tau_1 N & \tau_2 N & \tau_3 N \\
\text{(actual } \tau_0 N) & \text{(actual } \tau_1 N) & \text{(actual } \tau_2 N) & \text{(actual } \tau_3 N) \\
\end{array} \]

P.V. of (estimated stream) = \( \tau_0 N d_{01} + \tau_1 N d_{02} + \tau_2 N d_{03} + \tau_3 N d_{04} \) = \( N (\tau_0 d_{01} + \tau_1 d_{02} + \tau_2 d_{03} + \tau_3 d_{04}) \)

= \( N \sum_{i=0}^{3} \tau_i d_{0,i+1} \leftarrow = N (1 - d_{04}) \)

where: \( d_{01} = \frac{1}{1 + \tau_0} \), \( d_{02} = \frac{1}{(1 + \tau_0)(1 + \tau_1)} = d_{11} d_{12} \)

\( d_{03} = \frac{1}{(1 + \tau_0)(1 + \tau_1)(1 + \tau_2)} = d_{11} d_{12} d_{13} \)

\( d_{04} = \frac{1}{(1 + \tau_0)(1 + \tau_1)(1 + \tau_2)(1 + \tau_3)} \)

\[ \text{Lemma: if } d_{0,i+1} = \frac{1}{(1 + \tau_0)(1 + \tau_1) \cdots (1 + \tau_i)} \quad \text{then } \sum_{i=0}^{M-1} \tau_i d_{0,i+1} = 1 - d_{0M} \quad \text{for } M = 1, 2, \ldots \]
Recall: A pays B \( r \cdot N\left( \sum_{i=1}^{4} d_{0i} \right) \).

Choose \( r \) such that

\[
r \cdot N\left( \sum_{i=1}^{4} d_{0i} \right) = N\left( 1 - d_{04} \right).
\]