12.3 Forward Prices
Assumptions:

1. no transaction costs
2. assets can be divided arbitrarily
3. no storage costs (for now)
4. can sell asset short
Claim: \( F = \frac{S}{d(0,T)} \leftarrow \text{spot price} \)
\( \leq \text{discount factor: } d(0,T) < 1 \)

CASE 1
\( F > \frac{S}{d(0,T)} \)

- "Long" the asset: borrow \( S \)
- buy asset for \( S \)
- arrange a forward contract to sell asset for \( F \) at \( t = T \).

at \( t = 0 \): no initial outlay

at \( t = T \): exercise the forward contract: deliver the asset and receive \( F \).
- repay loan: pay \( \frac{S}{d(0,T)} \).

profit at \( t = T \) is
\[ F - \frac{S}{d(0,T)} > 0 : \text{arbitrage (type B)} \]

\( \therefore \) if no arbitrage,

\[ F - \frac{S}{d(0,T)} \leq 0. \]

CASE 2

\( F < \frac{S}{d(0,T)} \)

- "Short" the asset: borrow the asset from someone who owns it but plans to store it.
- sell the asset for \( S \), put funds \( S \) in bank.
- arrange a forward contract to buy asset for \( F \) at \( t = T \).

at \( t = T \): withdraw \( \frac{S}{d(0,T)} \) from bank.

Exercise the forward contract to buy the asset for \( F \); return asset to owner.

Profit at \( t = T \) is
\[ \frac{S}{d(0,T)} - F > 0 : \text{arbitrage} \]

\( \therefore \) if no arb., \( F \geq \frac{S}{d(0,T)} \).
Discount factors

Notation $d(a, b)$ converts cash at time $b$, to cash at time $a$:

$\text{cash at time } a = \text{cash at time } b \times d(a, b)$.

Normally, $a < b$, $d(a, b) < 1$

Notation $d_{ij} = d(i\Delta t, j\Delta t)$

Example $\Delta t = \frac{1}{2} \text{ year}$:

$d_{01} = d(0, \frac{1}{2}\Delta t)$: converts cash at $t = \frac{1}{2}\Delta \text{ year}$

to cash at $t = 0$

$d_{02} = d(0, 2\Delta t)$

Identity: $d_{ij} = d_{ik} \times d_{kj}$
Consider a 5-year bond, $1000 face value, pays $70 coupon once per year.

<table>
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<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5 year</th>
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<tr>
<td>face value payoff</td>
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<td></td>
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<td></td>
<td>1000</td>
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</table>

Present value using discount factors:

\[
P.V. = 70 \cdot d_{0.1} + 70 \cdot d_{0.2} + 70 \cdot d_{0.3} + 70 \cdot d_{0.4} + 70 \cdot d_{0.5} + 1000 \cdot d_{0.5}
\]

(If \( S \) = spot price of bond, \( S = P.V. \))
Consider the same bond, but purchased 2 years from now.

\[ \begin{array}{cccccc}
\text{now=0} & 1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
70 & 70 & 70 & \phantom{0} & \phantom{0} \\
1000 & \\
\end{array} \]

\[ F_2 = \text{forward price for delivery in 2 years, of bond.} \]

\[ = \text{value 2 years from now, of income stream shown} \]

\[ = 70 \, d_{23} + 70 \, d_{24} + 70 \, d_{25} + 1000 \, d_{25} \]

Recall: \( d_{ij} = d_{ij}, \quad i \leq k \leq j \).

Mult. by \( d_{02} \)

\[ F_2 \, d_{02} = 70 \, d_{02} \, d_{23} + 70 \, d_{02} \, d_{24} + (70 + 1000) \, d_{02} \, d_{25} \]

\[ = 70 \, d_{03} + 70 \, d_{04} + (70 + 1000) \, d_{05} \]
But: \[ S = 70d_{o1} + 70d_{o2} + 70d_{o3} + 70d_{o4} + (70 + 1000)d_{o5} \]

\[ F_2\overline{d_{o2}}. \]

\[ S - F_2\overline{d_{o2}} = 70d_{o1} + 70d_{o2}. \]

\[ F_2 = \frac{1}{\overline{d_{o2}}} (S - 70d_{o1} - 70d_{o2}) \]

\[ = \frac{S}{\overline{d_{o2}}} - 70 \frac{d_{o1}}{\overline{d_{o2}}} - 70 \frac{d_{o2}}{\overline{d_{o2}}} \quad \overline{d_{o2}} = d_{o1}d_{12} \]

\[ \therefore \frac{d_{o1}}{\overline{d_{o2}}} = \frac{1}{d_{12}}. \]

\[ F_2 = \frac{S}{\overline{d_{o2}}} - 70 \cdot \frac{1}{d_{12}} - 70. \]