12.3 Forward Prices
Let $F_m = \text{forward price of bond, to be delivered at } t = \text{MAT in future.}$

Assume bond still running at $M+1.$

Let coupons be $c_1, c_2, \ldots, c_{M-1}, c_M, c_{M+1}, \ldots$

- to original owner
- to new owner.

At $t = \text{MAT, new owner pays original owner } F_m,$ and takes bond.

P.V. of $F_m,$ is $F_m + \text{dom.}$

(P.V. of coupons $c_1 \ldots c_m,$ is: $c_1 \times d_{01} + c_2 \times d_{02} + \ldots + c_m \times \text{dom}$)

Spot price $S = \text{P.V. of stream} = F_m + c_1 \times d_{01} + c_2 \times d_{02} + \ldots + c_m \times \text{dom.}$
Forward price $F_m = \frac{1}{d_{om}} (S - c_1 d_{o1} - c_2 d_{o2} - \cdots - c_m d_{om})$.

Can simplify if identities $d_{ij} = d_{ik} d_{kj}$ available.

Ex. Cost: Here, $c_1, c_2, \ldots, c_m$ are positive asset with carrying costs.

Ex. Gold: Suppose $S = $1236/ounce.

Suppose storage cost is $8 per ounce per year:

- payable quarterly in advance.

Suppose interest rate is $r = 2\%$, compounded quarterly.

What is the forward price of gold, for delivery in 1 year?
\( \Delta t = 0.25 \text{ year} \)

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\text{Cost of storage} & -2 & -2 & -2 & -2 & -2 + F_4
\end{array}
\]

Income from sale of gold 1 yr in future:

\[ \text{P.V.} = \text{P.V. of income stream} \]
\[ = -2 -2d_{01} -2d_{02} -2d_{03} + F_4 d_{04} \]

Spot price is \( S = \$1236 \).

Since \( S = \text{P.V. of income stream} \),

\[ S = -2(1 + d_{01} + d_{02} + d_{03}) + F_4 d_{04} \]

\[ \Rightarrow F_4 = S + 2(1 + d_{01} + d_{02} + d_{03}) \]

More generally:

\[ S = - \sum_{k=0}^{M-1} c_k d_{0k} + F_M d_{0M}; \quad F_M = \frac{1}{d_{0M}} \left( S + \sum_{k=0}^{M-1} c_k d_{0k} \right) \]
<table>
<thead>
<tr>
<th>Delta t</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25000</td>
<td>0.02000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>time t</th>
<th>0.00000</th>
<th>0.25000</th>
<th>0.50000</th>
<th>0.75000</th>
<th>1.00000</th>
</tr>
</thead>
<tbody>
<tr>
<td>d00</td>
<td>1.00000</td>
<td>0.99502</td>
<td>0.99007</td>
<td>0.98515</td>
<td>0.98025</td>
</tr>
<tr>
<td>d01</td>
<td>0.99502</td>
<td>0.99007</td>
<td>0.98515</td>
<td>0.98025</td>
<td></td>
</tr>
<tr>
<td>d02</td>
<td>0.99007</td>
<td>0.98515</td>
<td>0.98025</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d03</td>
<td>0.98515</td>
<td>0.98025</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d04</td>
<td>0.98025</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| c0     | -2.00000 | -2.00000 | -2.00000 | -2.00000 |
| c1     | -2.00000 | -2.00000 | -2.00000 | -2.00000 |
| c2     | -2.00000 | -2.00000 | -2.00000 | -2.00000 |
| c3     | -2.00000 | -2.00000 | -2.00000 | -2.00000 |

\[ \text{sum}(c_k \cdot d_{0k}, k=0..3) = -7.94050 \]
\[ \frac{\text{sum}(c_k \cdot d_{0k}, k=0..3)}{d_{04}} = -8.10050 \]

<table>
<thead>
<tr>
<th>S</th>
<th>1236.00000</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{S}{d_{04}} ]</td>
<td>1260.90602</td>
</tr>
</tbody>
</table>

\[ S - \text{sum}(c_k \cdot d_{0k}, k=0..3) = 1243.94050 \]
\[ F_4 \]
\[ 1243.94050 \]
\[ F_4 \cdot d_{04} \]
\[ 1243.94050 \]