12.10 The Minimum-Variance Hedge
Recall 13.4

Value at time $t$ of holding a forwards contract is

$$V(t) = (F(t) - F(0)) d(t, T)$$

Price written into contracts at time $t$ for expiration/delivery $T$

A contract arranged at $t=0$ has value

$$(F(T) - F(t))$$ at delivery.

Now use future for hedging.

Treat future like forward.
Last day:

Planning to spend cash at time $T$
to purchase $W$ units of a commodity at $S(T)$, spot price.

$x = -W \cdot S(T)$.

To hedge: go long $h$ units of a future.

$F(0) =$ price now, $F(T) =$ price at time $T$
within the contract.

Gain or loss from future will be

$$(F(T) - F(0)) \cdot h.$$ 

Net at delivery:

$$y = x + (F(T) - F(0)) \cdot h$$

\[\text{what will be spent on $W$ units of $S(T)$} \]

\[\text{gain or loss from $h$ units of future} \]

\[\text{random: } x, F(T) \]

\[S(T), y \]

\[\text{if available, } F(T) = S(T) ? \]
Identities: \( X, Y \) random; \( a, b, c \) const.

\[
\text{var} (X+a) = \text{var} (X) \quad \text{var} (X+a) = \mathbb{E} [(X+a - \bar{X} + \bar{Y})^2] \\
= \mathbb{E} [(X - \bar{X})^2] = \text{var} (X)
\]

\[
\text{var} (bX) = b^2 \text{var} (X).
\]

\[
\text{var} (X+Y) = \mathbb{E} [(X+Y - (\bar{X}+\bar{Y}))^2] \\
= \mathbb{E} [((X-\bar{X})+(Y-\bar{Y}))^2] \\
= \mathbb{E} [(X-\bar{X})^2] + 2 \mathbb{E} [(X-\bar{X})(Y-\bar{Y})] + \mathbb{E} [(Y-\bar{Y})^2] \\
= \text{var} (X) + 2 \text{cov} (X, Y) + \text{var} (Y).
\]

\[
\text{cov} (cX, Y) = \text{cov} (X, cY) = c \text{cov} (X, Y).
\]

\[
\text{cov} (X+a, Y+b) = \text{cov} (X, Y)
\]
Model hedging grapefruit juice with orange juice futures.

Suppose

\[ S = a_0 + a_1 z_1 + a_2 z_2 \quad \text{where} \quad a_0, a_1, a_2 \quad \text{constant}, \]

\[ F = b_0 + b_1 z_1, \]

\[ z_1, z_2 \quad \text{random variables:} \]

\[ E[z_1] = \bar{z}_1 = 0, \quad E[z_2] = \bar{z}_2 = 0, \]

\[ \text{var}(z_1) = 1, \quad \text{var}(z_2) = 1, \]

\[ \text{cov}(z_1, z_2) = 0. \]

Explanation:

Growing seasons differ:

- oranges:
  \[ z_1 \]
- grapefruit:
  \[ z_2 \]

\[ S = \text{price of grapefruit juice at } T \]

\[ F = \text{price of orange juice at } T. \]
Consider \( y(h) = -WS(T) + h(F(T) - F(0)) \) const.

\[
\text{var}(y(h)) = \text{var}(-WS(T) + hF(T) - hF(0))
\]

\[
= \text{var}(-WS(T) + hF(T))
\]

\[
= \text{var}(-WS(T)) + 2\text{cov}(-WS(T), hF(T)) + \text{var}(hF(T))
\]

\[
= W^2\text{var}(S(T)) - 2Wh\text{cov}(S(T), F(T)) + h^2\text{var}(F(T)).
\]

Minimize: set \( \frac{\partial}{\partial h} \text{var}(y(h)) = 0. \)

\[
\frac{\partial}{\partial h} \text{var} y(h) = 0 - 2W\text{cov}(S(T), F(T)) + 2h\text{var}(F(T)).
\]

\( \Rightarrow h = h_{\text{min}} = \frac{W\text{cov}(S(T), F(T))}{\text{var}(F(T))} \)

Claim:

\[
\text{var}(y(h_{\text{min}})) = W^2 \left[ \frac{\text{var}(S(T)) - \frac{[\text{cov}(S(T), F(T))]^2}{\text{var}(F(T))}}{\text{var}(F(T))} \right]
\]
\[ \text{var } F = \text{var } (b_0 + b_1 y_1) = \text{var } (b_1 y_1) = b_1 \text{ var } (y_1) = b_1^2. \]

\[ \text{cov } (S, F) = \text{cov } (a_0 + a_1 S_1 + a_2 S_2, b_0 + b_1 S_1) \]
\[ = \text{cov } (a_1 S_1 + a_2 S_2, b_1 S_1) \]
\[ = a_1 b_1 \text{ cov } (S_1, S_1) + a_2 b_1 \text{ cov } (S_2, S_1) \]
\[ = a_1 b_1 \text{ var } (S_1) + a_2 b_1 \text{ cov } (S_2, S_1) \]
\[ = a_1 b_1 \frac{\text{var } (S_1)}{1} + a_2 b_1 \text{ cov } (S_2, S_1) \]
\[ = 0 \text{ if weather for two periods unrelated.} \]

\[ \text{var } (y(\text{min})) = w^2 \left[ \text{var } (S(T)) - \frac{\text{cov } (S(T), F(T))^2}{\text{var } (F(T))} \right] \]
\[ = w^2 \left[ a^2 + a_2^2 - \frac{(a_1 b_1)^2}{b_1^2} \right] = w^2 a_2^2. \]

\( y(\text{min}) \) remains a random variable if \( a_2 \neq 0. \)

\( \text{basis risk if } a_2 \neq 0 \).