11.5 Utility Functions and the Mean-Variance Criterion
Combine utility with Markowitz problem
(no risk free; 3 risky assets
return rates \( r_1, r_2, r_3 \))
random vars
\[ \mathbb{E}[r_i] = \bar{r}_i, \quad \mathbb{E}[r_2] = \bar{r}_2, \quad \mathbb{E}[r_3] = \bar{r}_3, \]
\[ \text{var}(r_1) = \sigma_1^2, \quad \text{var}(r_2) = \sigma_2^2, \quad \text{etc.} \]
\[ \text{cov}(r_i, r_j) = \sigma_{ij} \quad \text{given.} \]

Suppose portfolio with initial value \( W_0 \).

After \( t \) yr, \( y = (1 + r)W_0 \) where \( r \), \( n \), \( u \) is return rate
for portfolio.

Portfolio weights \( w_1, w_2, w_3 \) such that \( w_1 + w_2 + w_3 = 1 \)
and \( w_1 \geq 0, w_2 \geq 0, w_3 \geq 0 \).
(as shorting)
Consider utility $U[y]$, \[ y = (1 + \bar{\eta}) w_0 \]
\[ = (1 + \overline{w_1 \eta_1} + \overline{w_2 \eta_2} + \overline{w_3 \eta_3}) w_0. \]

One problem to consider is

maximize \[ E[U(y)] , \]

with constraints

\[ w_1 + w_2 + w_3 = 1 , \]

\[ E[y] = \bar{y} , \]

where \( \bar{y} \) is given (in book, M).

But:

\[ E[y] = E[(1 + \bar{\eta}) w_0] = (1 + \bar{\eta}) w_0 \]

where

\[ \bar{\eta} = \bar{E}[\eta] \]

\[ = \bar{E}[w_1 \eta_1 + w_2 \eta_2 + w_3 \eta_3] \]

\[ = w_1 \bar{\eta}_1 + w_2 \bar{\eta}_2 + w_3 \bar{\eta}_3 . \]

The constraint \( E[y] = \bar{y} \)

is equivalent to:

\[ \bar{\eta} = \overline{w_1 \eta_1} + w_2 \bar{\eta}_2 + w_3 \bar{\eta}_3 = \bar{\eta} , \]

where

\[ \bar{\eta} = \frac{\bar{y} - w_0}{w_0} - 1 . \]
Special case: \( U(y) = ay - \frac{b}{2} y^2 \)

restrict to \( y \) where \( U'(y) > 0 \)

\[ E[U(y)] = a E[y] - \frac{b}{2} E[y^2], \]

\( y \) is r.v.

For r.v. \( y \), \( \text{var}(y) = E[y^2] - \bar{y}^2 \)

\( (\text{var}(y) = E[(y-\bar{y})^2]) \)

\[ E[U(y)] = a \bar{y} - \frac{b}{2} \left( \bar{y}^2 + \text{var}(y) \right), \]

and constraints, \( w_1 + w_2 + w_3 = 1 \),

\[ w_1 \bar{x}_1 + w_2 \bar{x}_2 + w_3 \bar{x}_3 = \bar{x} = \left( \frac{y}{w_0} - 1 \right) \]

So: \( \bar{y} \) is being held constant; \( ay - \frac{b}{2} \bar{y}^2 \) is held constant.

max \( E[U(y)] \) is obtained by minimizing \( \text{var}(y) \).

Note: how to do!
Another special case:

If returns are normal random vars,
then (claim) Markowitz approach (min var)
and utility function approach
both give same soln, for any risk-averse util. fn.

Let $U$ be a util. fn:

suppose $y$ is normal, mean $M$, $\text{var}(y) = 6^2$.

Notice $M, 6^2$ completely determine the prob. distribution.
then $M, 6^2$ completely determine $E[U(y)]$:

$$E[U(y)] = f(M, 6).$$

$f$ is an increasing fn of $M$, a decreasing fn of $6$.

Portfolio is a linear comb. of normal r. v.'s,
therefore normal r. v.; so get an $f$ for portfolio

For fixed $y = M$, maximize $f(M, 6)$ by minimizing $\delta = 6(w_1, w_2, w_3)$
take min $6(w_1, w_2, w_3)$, subject to constraints.
Links visited in class

Feasible set for Markowitz problem sigma_rbar_three_asset_noshorting. mp4, html