11.2 Utility Functions
Utility example: Amount $W_c$.

Can put money in a bank at a fixed interest rate.

After 1 year, payoff = $10K$.

Consider another possible investment of the $W_c$.

a risky scheme: $X$:

$X = r.v. \text{ with payoff } = \begin{cases} 
$30K \text{ with prob. } \frac{1}{3} \\
$4K \text{ with prob. } \frac{2}{3} 
\end{cases}$

Suppose your utility function is $U(X) = \sqrt{X}$.

Which investment do you choose?

Utility of bank: $U(10K) = U(10^4) = 100$.

Compare with expected utility of risky investment.

$E [U(X)] = U(30K) \frac{1}{3} + U(4K) \frac{2}{3} < 99.9$

Since $100 > 99.9$, choose bank.
Expected

The utility of the risky investment is $E [U(x)] = 99.9$.

What would the investor accept as a certain payoff, instead of the risky investment?

Let $C = \text{certain payoff: } U(C) = 99.9$.

\[
\sqrt{C} = 99.9
\]

\[
C = (99.9)^2 = 9980 = 9.98K
\]

This $C$ is called the certainty equivalent of $X$. 
Example functions $U(x)$

$U(x) = -e^{-ax}$, $a > 0$.

$U(x) < 0$ is o.k.: what’s important is comparing possible outcomes.

Basic property: $\forall x < y \Rightarrow U(x) < U(y)$

Equivalently,

$U(x) = 1 - e^{-ax}$

$U(x) = \ln x$
\( U(x) = x^b, \quad 0 < b < 1. \)

\[ U(x) = x - bx^2, \quad b > 0 \]

and \( 0 \leq x \leq \frac{1}{2b} \)

**Properties of Utility Function:**

If \( x < y \), then \( U(x) < U(y) \). \( U \) is increasing.

Can assume \( U \) is continuous.
Usually, given random variables $X$ and $Y$, payoffs from two possible investments, want to use $U$ to compare $E[U(X)]$ and $E[U(Y)]$.

In order to choose one of $X$ or $Y$, the investment with the larger expected utility of returns, payoff.

Simplest utility is $U(x) = x$, an example of risk-neutral utility. Each extra dollar, produces the same increase in utility:

$$U(x+1) = U(x) + \left[U(x+1) - U(x)\right] = U(x) + 1.$$

Also:

$$U(x+1) = U(x) + U'(x)(x+1) - x = U(x) + U'(x).$$
In general, expect marginal increase in utility to decrease at higher wealth levels $x$.

Slope $u'(x)$ decreases as $x$ increases, leading to a concave function $u(x)$.

The graph of $u(x)$ is concave if, for every pair $x, y$ with $x < y$,

the points on the line from $(x, u(x))$ to $(y, u(y))$ fall below the graph!
Consider an investment that has payoff

\[ X = \begin{cases} \mathcal{X} \text{ with prob. } 1 - \alpha \\ \mathcal{Y} \text{ with prob. } \alpha \end{cases} \]

some known \( \alpha, 0 < \alpha < 1 \).

\[ E[X] = (1 - \alpha) \mathcal{X} + \alpha \mathcal{Y}. \]

Decisions to be made based on \( E[U(X)] \) (versus \( E[U(Y)] \))

\[ E[U(X)] = (1 - \alpha)U(\mathcal{X}) + \alpha U(\mathcal{Y}). \]

Consider an alternative investment

without risk, that pays off \( (1 - \alpha) \mathcal{X} + \alpha \mathcal{Y} = \mathcal{X}_0 \).

To the investor, this certain payoff has utility \( U((1 - \alpha) \mathcal{X} + \alpha \mathcal{Y}) \).

Since \( U \) is concave,

\[ (1 - \alpha)U(\mathcal{X}) + \alpha U(\mathcal{Y}) < U(\mathcal{X}_0). \]

\[ E[U(X)] < U(\mathcal{X}_0): \]

utility of risk-free payoff in amount \( \mathcal{X}_\alpha = E[X] \).

\( \mathcal{X}_\alpha \) is risk-averse

\( \mathcal{X}_\alpha = E[X] \)