11.2 Utility Functions
Utility: Read §11.1-11.3.

St. Petersburg paradox: Nicholas Bernoulli

Consider the game: Toss a fair coin, until head comes up.

If heads on first toss, payoff = $1: \text{prob. } = \frac{1}{2}.

If heads on second toss, payoff = $2: \text{prob. } = \frac{1}{4}.

If heads on third toss, payoff = $4: \text{prob. } = \frac{1}{8}.

" fourth "     = $8: \text{prob } = \frac{1}{16}.

" kth "        = $2^{k-1}: \text{prob } = \frac{1}{2^k}.

What are you willing to pay, to participate in this game?

If you think in terms of expected payoff,
you would pay any amount < expected payoff.

But

\[
\text{Expected payoff } = \frac{1}{2} \times \frac{1}{2} + \frac{2}{4} \times \frac{1}{2} + \frac{4}{8} \times \frac{1}{2} + \frac{8}{16} \times \frac{1}{2} + \cdots
\]

\[= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty
\]
Daniel Bernoulli (cousin)

Value of game to player, as utility it yields.

\[ U(x) = \text{utility of wealth level } x. \]

Instead of looking at \( \sum_{k} p_k x_k \frac{E(X)}{\text{prob. of outcome } x_k} \),

look instead at \( \sum_{k} p_k U(x_k) = E(U(X)) \).

D. B. suggested \( U(x) = \ln x \).

For this choice,

\[ \sum_{k=1}^{\infty} p_k x_k U(x_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} \ln (2^{k-1}) \]

\[ = \left( \sum_{k=1}^{\infty} \frac{k-1}{2^k} \right) \ln 2 = (1) \ln 2. \]

But:

\[ U(2) = \ln 2 = \sum_{k=1}^{\infty} p_k U(x_k) = E(U(X)) \]

Suggest game is worth \$2.

\$2 is the "certainty equivalent" of the game.