9.2 Discrete hedging
Why interested in $\Delta$?

9.2 Discrete Hedging:

Can use "Delta Hedging" to decrease sensitivity of a portfolio with respect to changes in underlying $S$.

Suppose portfolio consists of 1 call option, price $C(s,t)$, and $k$ units of $S$.

$$\Pi = C(s,t) + kS.$$ 

The $\Delta$ for portfolio is

$$\Pi_\Delta = \frac{\partial}{\partial S} (C(s,t) + kS) = \frac{\partial C}{\partial S}(s,t) + k.$$

Want: $\Pi_\Delta = 0$

At $t = t_0$, $S = S_0$, make $k = -C_\Delta(S_0, t_0)$.

For each option we buy, we short $C_\Delta$ units of asset.
Once we've bought or sold at \( t = t_0, S = S_0 \) as \( t \) increases, the \( \Pi_\Delta \) "drifts away" from 0.

From time to time, need to rebalance.

- Change the relative amounts of stock/call options to restore \( \Pi_\Delta = 0 \).

\( \Gamma \) measures sensitivity of \( \Delta \), with \( S \):

\[
\Pi_\Gamma = \frac{\partial}{\partial S} (\Pi_\Delta) = \frac{\partial^2}{\partial S^2} (C_\Delta(S,t) + \Delta)
\]

\[= C_\Gamma(S,t) + \Gamma, \quad C_\Gamma = \frac{\partial^2 C}{\partial S^2} .\]

If \( |\Pi_\Gamma| = |C_\Gamma| \ll 1 \), then \( \Pi_\Delta \) changes slowly with \( S \).

i.e., less need to rebalance as \( S \) changes.
Call: make a bet that underlyings will be greater than $K$, at expiration $T = T$.

Payoff $X^\text{con}(S) = \begin{cases} A & \text{if } S > K \\ A/2 & \text{if } S = K \\ 0 & \text{if } S < K \end{cases}$ a probability event.

To evaluate cash-or-nothing call value:

$$C^\text{con} = e^{-r(T-t)} \mathbb{E} [X^\text{con}(S(T))].$$

$$= e^{-r(T-t)} \int_{y=0}^{\infty} \Lambda(y) e^{-\left(\ln y - \ln S - (r - \frac{1}{2} \sigma^2)(T-t)\right)^2 / 2\sigma^2(T-t)} \frac{dy}{\sqrt{2\pi} \sigma \sqrt{T-t}}.$$

Or:

Note that $\Lambda^\text{con} = A \times \frac{2}{S} \Lambda^\text{euro} = A \times \frac{1}{S} \max(S-K, 0)$.

As $t \to T^-$, $C^\text{con} \approx A \frac{2}{S} \Lambda^\text{euro} (S, T) = A C^*_\Delta$.
Cash-or-nothing Put:

Make a bet that underlying will be less than $K$:

\[
\text{Payoff: } \Lambda_{\text{con}} = \begin{cases} 
0 & \text{if } S > K \\
\frac{K}{2} & \text{if } S = K \\
A & \text{if } S < K.
\end{cases}
\]

Can use integral rep. to find $\Lambda_{\text{con}}$,

or note that

\[
\Lambda_{\text{con}} = -A \frac{d}{dS} \Lambda_{\text{euro}} = -A \frac{d}{dS} \left( \max(S-K, 0) \right)
\]

Claim:

As $t \to T^-$, $P_{\text{con}} \approx -A \frac{d}{dS} P_{\text{euro}} = -AP_{\Delta}$.