22.0 Monte Carlo Part III: variance reduction by control variates
Control variate schemes

Suppose we wish to compute \( E[f(x)] \), \( X \) a random var.

Suppose we can find \( g = g(x) \) such that

1. \( g(x) \) correlates with \( f(x) \): \( \text{cov}(f(x), g(x)) \geq 0 \)
   and (if possible)
   \[ \text{var}(f(x) - g(x)) \text{ is "small"} \]

2. We know \( \bar{g} = E[g(x)] \), exactly.

Write \( f = g + (f - g) \)

so

\[ E[f(x)] = E[g(x)] + E[f(x) - g(x)] \]

and (M.C.) \( E[f(x)] \approx \frac{1}{M} \sum_{i=1}^{M} \left( \bar{g} + f(x_i) - g(x_i) \right) \)
we call \( g(x) \) a control variate.

The 95\% c.i for the approx above,

\[
\text{has radius } \frac{1.96}{\sqrt{\text{var} (\xi(x) - g(x))}}.
\]

\[
\xi(x) = \frac{1}{1 + x}, \quad X \text{ uniform } [0, 1]
\]

\[
\begin{align*}
g &= 1 - \frac{x}{2} \\
\mathbb{E}[g] &= \int_0^1 (1 - \frac{x}{2}) \, dx = \frac{3}{4} \\
\bar{g} &= \frac{3}{4}.
\end{align*}
\]

Scheme to approx. \( \mathbb{E}[\xi(X)] \approx \frac{1}{M} \sum_{i=1}^{M} \frac{3}{4} + \frac{1}{1 + X_i} - (1 - \frac{X_i}{2}) \)

where \( X_i \) : pseudo random, uniform on \([0, 1]\).
\[ \text{var} \left( \ell(x) + \Theta (\bar{g} - g(x)) \right) \]
\[ = E \left[ (\ell(x) - \bar{\ell}) + \Theta (\bar{g} - g(x)) \right]^2 \]
\[ = E \left[ (\ell(x) - \bar{\ell})^2 \right] + 2 E \left[ (\ell(x) - \bar{\ell}) \Theta (\bar{g} - g(x)) \right] \]
\[ + \Theta^2 E \left[ (\bar{g} - g(x))^2 \right] \]
\[ = \text{var} \ell(x) + 2 \Theta \text{cov} (\ell(x), -g(x)) + \Theta^2 \text{var} (g(x)). \]

Take \( \frac{\partial}{\partial \Theta} \), set to 0 to minimize: find

\[ \Theta_{\text{min}} = \frac{\text{cov}(\ell(x), g(x))}{\text{var}(g(x))}; \text{ apply M.C. to } \ell(x) + \Theta_{\text{min}} (\bar{g} - g(x)) \]

Is this pair? \[ \text{cov} (\ell, g) = E \left[ (\ell - \bar{\ell})(g - \bar{g}) \right] \]
\[ \text{need estimate of } \bar{\ell} \text{ to start} \]
minimum variance control variate scheme

As above: want to find $E[f(x)]$.

Suppose we find $E[g(x)]$ exactly: $\bar{g} = E[g(x)]$

where $\text{cov}(f(x), g(x)) \geq 0$.

Consider $f + \theta (\bar{g} - g(x))$:

Then

$$E[f(x) + \theta (\bar{g} - g(x))]$$

$$= E[f(x)] + \theta E[\bar{g} - g(x)]$$

Wish to minimize the variance of $f(x) + \theta (\bar{g} - g(x))$. 