19.2 Barrier options
Amer/Euro Put & Call are "Vanilla" anything else "exotic"

Barrier Options:
Consider "down and out" barrier B.
Call option becomes worthless if asset S falls below B.

Payoff \( \Delta(S(T)) = \max(S(T)-K, 0) \); Euro.
The analytic solution of the B-S PDE

\[ C_{d_0}(s,t) = \begin{cases} 
C_{\text{eu}}(s,t) - \left( \frac{s}{B} \right)^{1 - \frac{2\eta}{\rho^2}} C_{\text{eu}} \left( \frac{B^2}{s}, t \right) & s \geq B \\
0 & s < B 
\end{cases} \]

where \( C_{\text{eu}}(s,t) \) = standard B-S formula

satisfies

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C_{\text{eu}}(B,t) - \left( \frac{B}{B} \right)^{1 - \frac{2\eta}{\rho^2}} C_{\text{eu}} \left( \frac{B^2}{B}, t \right) & s \geq B \\
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\end{cases} \]

\[ C_{d_0}(B,t) = 0 \]
Check payoff as \( t \to T^- \):

\[ d_1, d_2 \text{ terms in } C^\infty \left( \frac{B^2}{s}, t \right) \]

\[ d_{1,2} = \frac{\ln \left( \frac{B^2}{s} \right) + \left( r + \frac{\sigma^2}{2} \right)(T-t) - \ln K}{\sigma \sqrt{T-t}} \]

As \( t \to T^- \), if \( s(t) \to s(T) > B \),

then \( \ln \left( \frac{B^2}{sk} \right) \to \ln \left( \frac{B}{s(T)} \cdot \frac{B}{K} \right) < 0 \)

\( < 1 \times 1 < 1 \)

Both \( d_1, d_2 \to -\infty \), \( N(d_1) \to 0 \), \( N(d_2) \to 0 \)

\[ \Rightarrow C \left( \frac{B^2}{s}, t \right) \to 0. \]

\[ C_{d_0}(S, t) \to C \left( S(T), T \right) = \Delta(S(T)) = \max(S(T) - K, 0). \]

If \( S(T) < B \), \( C_{d_0} = 0 \) since "down-and-out" before \( T \).
Exercises 19.1/19.2 show that for any constant $X,$

$$
\left( \frac{S}{B} \right)^{1 - \frac{2\sigma^2}{6\sigma^2}} \exp \left( \frac{B^2}{S}, t \right) \text{ satisfies BS.}
$$

actually,

$$
\hat{V}(S, t) = S^{1 - \frac{2\sigma^2}{6\sigma^2}} V \left( \frac{X}{S}, t \right)
$$

de $V(S, t)$ satisfies BS, so does $\hat{V}(S, t).$

BS is linear homogeneous, so can mult. by $\left( \frac{1}{B} \right)^{1 - \frac{2\sigma^2}{6\sigma^2}}.$
There is an "issue":

The barrier option value depends not only on \((S,t)\), but also on the path from \((S_0,0)\) to \((S,t)\), which may have crossed the barrier.

![Path diagram]

The value is not given by a PDE:

formulas are not correct, but can serve as an upper bound.

More accurate: use Monte-Carlo with large \# of steps/runs.

Need to simulate entire path, \(\epsilon\) to see if cross barrier.

Can't use \(S(T) = S(0) e^{(\mu - \sigma^2/2)T + \sigma \epsilon \sqrt{T}}\), \(\epsilon \) std. normal

"one-step".