18.0 American options
American options.

Let \( P^A_m = P^A_m(S, t) \) = value of Amer. style put.

Then: \( P^A_m(S, t) \geq \Delta(S(t)) = \max(K - S, 0) \):

may exercise at any time \( t, \, 0 \leq t \leq T \).

(This is different from \( P^{Euro} \); see "big picture", H. Ch. 11. Figure 1)

Suppose \( P^A_m(S, t) > K - S \), for some \((S, t)\).

Then: same is true for nearby values,

and Black-Scholes assumptions hold locally

\[ \Rightarrow \text{B-S P.D.E. is satisfied}. \]
If $S$ very small, $P^{Am} \leq K - S$; \hspace{1cm} \Rightarrow \text{exercise early.} \\

$P^{Am} = K - S$

For fixed $t \in (0, \tau)$

\[\begin{array}{c}
\hline \\
0 \uparrow & S^* & K \downarrow \\
\text{exercise} & \text{early} & \text{don't exercise here.}
\end{array} \]

Can show: there is a unique $S^*(t)$

such that $P^{Am} > K - S$ if $S > S^*(t)$,

$P^{Am} \leq K - S$ if $S < S^*(t)$.

As $t$ varies, find a curve $S = S^*(t)$.

Also: \hspace{1cm} \lim_{t \to T^-} S^*(t) = K.
As a problem in PDE's.

On "do not" exercise region \( P > K - S, \)

\[
\frac{\partial P}{\partial t} + \rho S \frac{\partial P}{\partial S} + \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2} = rP \quad (B.S.)
\]

On exercise region,

\[ P = K - S. \]

For the \( P: \)

\[
\frac{\partial P}{\partial t} = 0, \quad \frac{\partial P}{\partial S} = -1, \quad \frac{\partial^2 P}{\partial S^2} = 0
\]

L.h.s of B.S. = \(- \rho S + 0\); r.h.s. of B.S. = \( r(K - S) \)

\[ = - \rho S + rK \]

L.h.s. < r.h.s.

Combining both cases,

1. \( \frac{\partial P}{\partial t} + \rho S \frac{\partial P}{\partial S} + \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2} \leq rP \)

and

2. \( P \geq \Delta(S) = \max\{K - S, 0\} \)
3. \( P \) continuous

4. \( \frac{\partial P}{\partial S} \) continuous; solid "match" along \( S = S^*(t) \);

5. \( P(s, T) = \Lambda(s) = \max \{ K - S, 0 \} \)

Free boundary type problem:

on \( S = S^*(t) \), \( \sum \frac{\partial P}{\partial S}(S^*(t), t) = -1 \)

\( P(S^*(t), t) = K - S^*(t) \).