Dividends

Three ways-

1. Continuous dividend payments -
   easiest math, not bad, less realistic than discrete.

2. Discrete dividend payments -
   Issue: don't necessarily know dividend payout:
   2a: payout a fraction of the stock (asset)
   2b: payout a fixed value, known in advance
1. To modify for dividends: continuous dividend payments

Assume stock pays a fixed fraction $q$:

over time interval $dt$, it pays $qSdt$.

What happens to stock price?

$S$ falls by exactly the dividend payment.

Derive $B-S$ for option $f(S, t)$. 
\[
\frac{\partial f}{\partial t} + (r - q) S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.
\]

Only difference:

Try \( f(S, t) = e^{-q(t-t)} \psi(S, t) \).

\[
\psi_t = q e^{-q(t-t)} \psi + e^{-q(t-t)} \psi_S; \quad \psi_S = e^{-q(t-t)} \psi_S; \quad \psi_{SS} = e^{-q(t-t)} \psi_{SS}
\]

PDE becomes

\[
e^{-q(t-t)} \left[ q \psi + \psi_t + (r-q) S \psi_S + \frac{1}{2} \sigma^2 S^2 \psi_{SS} \right] = r e^{-q(t-t)} \psi
\]

or

\[
\psi_t + (r-q) S \psi_S + \frac{1}{2} \sigma^2 S^2 \psi_{SS} = (r-q) \psi
\]

which is the B-S PDE but with \( r-q \) instead of \( r \).

At expiration, \( \psi(S(T), T) = e^{\sigma^2 T/2} \mathbb{E}[f(S(T), T)] = \max(S(T) - K, 0) \).

\( \therefore \) \( f(S, t) \) given by B-S formula but with \( r-q \) as rate.
Let $\bar{x}_1, \bar{x}_2$ be B-S formulas for $d_1, d_2$ but with $n-g$ as risk.

\[
\bar{x}_{1,2} = \frac{\ln S + (r - q \pm \sigma^2/2)(T-t) - \ln K}{\sigma \sqrt{T-t}}
\]

then

\[
v(S, t) = SN(\bar{x}_1) - Ke^{-(n-g)(T-t)} N(\bar{x}_2)
\]

and

\[
C = \Phi(S, t) = e^{-\varphi(T-t)} N = Se^{-\varphi(T-t)} N(\bar{x}_1) - Ke^{-\varphi(T-t)} N(\bar{x}_2)
\]

Can show: worth less than $C_{\text{Euro}}(S, t)$

with $n$ as rate.

HW: same formula, but for put:

\[
Pd = Ke^{-\varphi(T-t)} N(-\bar{x}_2) - Se^{-\varphi(T-t)} N(-\bar{x}_1),
\]