R3. a) Apply Ito’s lemma to show that \( y = \sigma B_t + (r - \sigma^2/2)t \) satisfies the stochastic differential equation (S.D.E.)

\[
dy = (r - \sigma^2/2)dt + \sigma dB_t.
\]

b) Apply Ito’s lemma to show that \( S = S_0 e^{\sigma B_t + (r - \sigma^2/2)t} \) satisfies the S.D.E.

\[
dS = rS dt + \sigma S dB_t
\]

c) (538 only) A Wiki article gives the following form of Ito’s lemma: If \( dx = a dt + b dB_t \) and \( y = g(t, x) \), then

\[
dy = \left( a \frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 g}{\partial x^2} \right) dt + b \frac{\partial g}{\partial x} dB_t.
\]

The version of Ito’s lemma which appears in the textbook is:
If \( y = f(t, x) \), then

\[
dy = \frac{\partial f}{\partial t}(t, x) dt + \frac{\partial f}{\partial x}(t, x) dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) dx \ dx
\]

where the \( dx \ dx \) term is interpreted by using the identities \( dt \ dt = 0, dt \ dB_t = 0, dB_t \ dt = 0 \) and \( dB_t \ dB_t = dt \).
Show that the version in the Wiki article follows from the version in the textbook. Assume \( a \) and \( b \) are constants.
(a) \[ y = \sigma B_t + (r - \frac{\sigma^2}{2}) t \] satisfies \[ dy = (r - \frac{\sigma^2}{2}) dt + \sigma dB_t \]

\[ y = f(t, x) = \sigma x + (r - \frac{\sigma^2}{2}) t \] where \[ x = B_t \]

Itô: \[ dy = ft \, dt + fx \, dx + \frac{1}{2} f_{xx} \, dx \, dx \]

\[ ft = r - \frac{\sigma^2}{2}, \quad fx = \sigma, \quad f_{xx} = 0 \]

\[ dy = (r - \frac{\sigma^2}{2}) dt + \sigma dx + 0 = (r - \frac{\sigma^2}{2}) dt + \sigma dB_t \] since \[ x = B_t \]

\[ \text{So it's the soln.} \]

(b) \[ S = S_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2}) t} \Rightarrow ds = rS dt + \sigma S dB_t \]

\[ S = f(t, x) = S_0 e^{x}, \text{ where } x = \sigma B_t + (r - \frac{\sigma^2}{2}) t. \]

\[ ds = S_0 e^{x} dx + \frac{1}{2} S_0 e^{x} dx \, dx \]

since \[ dx = (r - \frac{1}{2} \sigma^2) dt + \sigma dB_t \]

\[ dx \, dx = \sigma^2 dt \]

\[ ds = S_0 e^{x} (r - \frac{1}{2} \sigma^2) dt + S_0 e^{x} \sigma dB_t + \frac{1}{2} S_0 \sigma^2 e^{x} dt \]

\[ = S_0 e^{x} r dt + S_0 e^{x} \sigma dB_t \]

\[ = r S dt + \sigma S dB_t \]
(C) In textbook:

\[ dy = \frac{\partial f}{\partial t}(t,x) dt + \frac{\partial f}{\partial x}(t,x) dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t,x) dt \, dx \, dx \]

\[ dx = a \, dt + b \, dB_t \]

So

\[ dy = \frac{\partial f}{\partial t}(t,x) dt + \frac{\partial f}{\partial x}(t,x) (a \, dt + b \, dB_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t,x) (a \, dt + b \, dB_t)^2 \]

\[ = \frac{\partial f}{\partial t}(t,x) dt + a \frac{\partial f}{\partial x}(t,x) dt + b \frac{\partial f}{\partial x}(t,x) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t,x) (a \, dt + 2ab \, dt \, dB_t + b^2 \, dB_t \, dB_t) \]

Because \( dt \, dt = 0, \; dt \, dB_t = 0, \; dB_t \, dB_t = dt \)

So we have

\[ dy = (a \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}) dt + b \frac{\partial f}{\partial x} dB_t + \frac{b^2}{2} \frac{\partial^2 f}{\partial x^2} dt \]

\[ = (a \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{b^2}{2} \frac{\partial^2 f}{\partial x^2}) dt + b \frac{\partial f}{\partial x} dB_t \]

for \( y = f(t,x) \).

So for \( y = g(t,x) \), we have:

\[ dy = (a \frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} + \frac{b^2}{2} \frac{\partial^2 g}{\partial x^2}) dt + b \frac{\partial g}{\partial x} dB_t \]

Which is the form from wiki.