4. Consider the heat equation \( u_t = cu_{xx} \), \( 0 < x < 1, \ t > 0 \) where \( c = \frac{1}{3} \), with initial condition \( u(x,0) = x(1-x) \) for \( 0 \leq x \leq 1 \) and boundary conditions \( u(0,t) = u(1,t) = 0, \ t \geq 0 \). **DO NOT solve for** \( u \)

a) Let \( m = 2, \ h = \frac{1}{m+1} \) and \( x_i = ih, \ i = 0, 1, ..., m + 1 \). Let \( k > 0 \) and \( t_j = jk, \ j = 0, 1, 2, ... \), and let \( w_{i,j} \approx u(x_i, t_j) \). Apply the difference approximations

\[
\frac{u(x, t + k) - u(x, t)}{k}, \quad \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2}
\]

to derive the forward difference in time, centered difference in space method

\[
w_{i,j+1} = w_{i,j} + \frac{ck}{h^2}(w_{i-1,j} - 2w_{i,j} + w_{i+1,j}) \quad \text{for} \quad i = 1, 2, \ j = 0, 1, 2, ...
\]

b) Determine the 2x2 matrix \( A \) such that

\[
\begin{bmatrix}
  w_{1,j+1} \\
  w_{2,j+1}
\end{bmatrix}
= A
\begin{bmatrix}
  w_{1,j} \\
  w_{2,j}
\end{bmatrix}, \ j = 0, 1, 2, ...
\]

Note that \( w_{0,j} = w_{3,j} = 0 \) for \( j = 0, 1, ... \)

c) Show that for initial data \( w_{i,0} = x_i(1-x_i) \), the solution obeys \( w_{1,j} = w_{2,j} \) for all \( j \).

d) Show that for \( k = \frac{1}{6} \), \( \lim_{j \to \infty} w_{1,j} = \lim_{j \to \infty} w_{2,j} = 0 \).

e) (538 only) Consider general \( k > 0 \). For what \( k \) is \( \lim_{j \to \infty} w_{1,j} = \lim_{j \to \infty} w_{2,j} = 0 \)? Explain why there is no inconsistency between the maximum value of \( k \) that you have found, and Theorem 8.2 which states that if \( k < \frac{h^2}{2c} \), the forward difference method is stable.
4. a) \( U_t (k) = \frac{w_{i+1} - w_i}{k} \) \hspace{1cm} \( U_{xx} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \)

\[ i = 0, 1, 2, \ldots \]

\[ j = 0, 1, 2, \ldots \]

\( U_t = c U_{xx} \)

\[ \Rightarrow \frac{w_{i+1} - w_i}{k} = c \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \]

\[ \Rightarrow (w_{i+1} - w_i) = \frac{ck}{h^2} (w_{i+1} - 2w_i + w_{i-1}) \]

\[ \Rightarrow w_{i+1} = w_i + \frac{ck}{h^2} (w_{i+1} - 2w_i + w_{i-1}) \]

\[ i = 0, 1, 2, \ldots \]

\[ j = 0, 1, 2, \ldots \]

(b) Let \( G = \frac{ck}{h^2} \)

\[ \Rightarrow w_{i+1} = w_i + 6(w_{i+1} - 2w_i + w_{i-1}) \]

\[ w_{0j} = w_j = 0 \]

\[ \Rightarrow w_{i+1} = 1 - 26w_i - 6w_j \]

\[ w_{i+1} = w_i + 6w_j - 26w_i \]

\[ w_{i+1} = (1 - 26)w_i + 6w_j \]

\[ w_{i+1} = w_i + 6w_j - 26w_i \]

\[ w_{i+1} = (1 - 26)w_i + 6w_j \]

\[ \begin{pmatrix} w_{i+1} \\ w_{j+1} \end{pmatrix} = \begin{pmatrix} 1 - 26 & 6 \\ 6 & 1 - 26 \end{pmatrix} \begin{pmatrix} w_i \\ w_j \end{pmatrix} \]

\[ i = 0, 1, 2, \ldots \]

\[ j = 0, 1, 2, \ldots \]

\[ \Rightarrow A = \begin{pmatrix} 1 - 26 & 6 \\ 6 & 1 - 26 \end{pmatrix} \]
\[ \text{Proof by induction} \]

\( W_{i_0} = X_i (1 - X_i) \Rightarrow W_{i_0} = X_i (1 - X_i) = \frac{1}{3} (1 - \frac{2}{3}) = \frac{1}{9} \]

\( W_{i_0} = X_i (1 - X_i) = \frac{2}{3} (1 - \frac{2}{3}) = \frac{2}{9} \)

\[ \Rightarrow W_{i_0} = W_{i_0} \]

\[ \text{\textbf{Suppose it is true for } j = k, \ W_{i_0} = W_{i_0} } \]

\( \text{when } j = k + 1 \)

\( W_{i, k+1} = (1 - 6) W_{i_k} + 6 W_{i_0} = (1 - 6) W_{i_k} + 6 W_{i_0} = (1 - 6) W_{i_k} \)

\( W_{i, k+1} = (1 - 6) W_{i_k} + 6 W_{i_0} = (1 - 6) W_{i_k} + 6 W_{i_0} = (1 - 6) W_{i_k} \)

\[ \Rightarrow W_{i, k+1} = W_{i, k+1} \]

\[ \Rightarrow W_{i, j} = W_{i, j} \text{ for all } j = 0, 1, 2, \ldots \]

\( \ldots \)

\[ \text{If } k = \frac{1}{6} \Rightarrow 6 = \frac{c k}{h^2} = \frac{1}{3} \cdot \frac{1}{6} \Rightarrow 1 - 6 = \frac{1}{2} \]

\[ \Rightarrow \text{By (c) we know } W_{i, j+1} = (1 - 6) W_{i, j} = \frac{1}{3} W_{i, j} \text{ and } W_{i_0} = \frac{1}{9} \]

\( \Rightarrow W_{i, j+1} = W_{i, j+1} = \frac{1}{3} W_{i, j} \)

\[ \Rightarrow \lim_{j \to \infty} W_{i, j+1} = \lim_{j \to \infty} W_{i, j+1} = \frac{1}{3} \left( \frac{2}{3} \right)^{j+1} = 0 \]

\( \text{From (d) we know in general, } W_{i, j+1} = W_{i, j+1} = (1 - \frac{c k}{h^2}) W_{i, j} \)

\[ \Rightarrow W_{i, j+1} = (1 - \frac{1}{3} \cdot \frac{2}{3}) \frac{2}{9} = (1 - 3 k) \frac{2}{9} = W_{i, j+1} \]

\[ \text{If } \lim_{j \to \infty} W_{i, j+1} = 0 \Rightarrow \lim_{j \to \infty} (1 - 3 k) = 0 \Rightarrow k < \frac{2}{3} \]

\[ \text{and } \frac{h^2}{2c} = \left( \frac{1}{3} \right) \frac{3}{2} = \frac{1}{6} \]

\[ \Rightarrow \frac{2}{3} > \frac{1}{6} \]

\[ \Rightarrow k < \frac{1}{6} \text{ for sure, no inconsistency} \]