2. We are given the values \( f_j = f(t_j) = f\left(\frac{j}{n}\right) \) \( j = 0,..,n-1 \) of a function \( f(t) \) which we know has the form

\[
f(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} (a_k \cos(2\pi kt) - b_k \sin(2\pi kt)) + \frac{a_{n/2}}{\sqrt{n}} \cos(n\pi t)
\]

We are designing a computer program to solve a differential equation involving \( f \). We will use a function \( \text{dft}(x) \) that, given a real vector \( x = (x_0, x_1, .., x_{n-1})^T \), returns a complex vector \( y = (y_0, y_1, .., y_{n-1})^T \) such that

\[
y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \omega^{jk}, \quad j = 0,..,n-1
\]

where \( \omega = e^{-i2\pi/n} \). The first goal is to use the function \( \text{dft}(x) \) to compute the coefficients \( a_k \) and \( b_k \) for \( f \). How would you store the values \( f_j \) in the array \( x \)? Give formulas for the coefficients \( a_k \) and \( b_k \) in terms of the components of the complex vector \( y = \text{dft}(x) \).

b) We wish to solve the differential equation

\[
X''(t) = f(t)
\]

where \( f \) is as above, to determine a function \( X(t) \) such that \( X(t+1) = X(t) \) for all \( t \) (\( X \) is 1-periodic.) Calculus tells us that \( X(t) \) must be of the form

\[
X(t) = C_0 + C_1 t + C_2 t^2 + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} [A_k \cos(2k\pi t) - B_k \sin(2k\pi t)] + \frac{A_{n/2}}{\sqrt{n}} \cos(n\pi t)
\]

Assume that \( a_0 = 0 \). Find values of the coefficients \( A_k, B_k, C_k \) such that \( X(t) \) satisfies the differential equation and is 1-periodic. Is the solution unique?

c) We will use a function \( \text{idft}(Y) \) that, given a complex vector \( Y = (Y_0, Y_1, .., Y_{n-1})^T \) with components such that \( Y_{n-k} = \overline{Y_k}, \quad k = 0,..,n-1 \), returns the inverse discrete fourier transform \( X \), the real vector with components

\[
X_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Y_k \omega^{-jk}, \quad j = 0,..,n-1,
\]

The goal is to use \( \text{idft}(Y) \) to compute the solution \( X(t) \) at the points \( t_j = \frac{j}{n}, \quad j = 0,..,n-1 \). What data would you store in the array \( Y \)? Give formulas for the values \( X(t_j) \) in terms of the elements of the vector \( X \).

d)(538 only) Show that if the condition \( a_0 = 0 \) in b) is not satisfied, then there is no 1-periodic solution \( X(t) \).
2. (a) Note that given \( y_k = \frac{1}{n} \sum_{j=0}^{n-1} x_j w^{jk} \), \( k = 0, \ldots, n-1 \) \( w = e^{-i\frac{2\pi}{n}} \),

Then \( y_j \) is the Fourier transforms of \( x_j \).

Because we have to determine \( \theta(\frac{n}{2} + \frac{1}{2}) \) to \( a \)'s and \( b \)'s.

Then I will put first \( \theta (\frac{n}{2} + \frac{1}{2}) f_j \) into \( x \) and do dft(\( x \)).

Then for \( A_k = \frac{Y_k + Y_{-k}}{2} \), \( B_k = \frac{Y_k - Y_{-k}}{2i} \), \( k = 0, \ldots, \frac{n}{2} \).

(b) \( x(t) = c_0 + c_1 t + c_2 t^2 + \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n}{2}-1} \left( A_k \cos k\pi t - B_k \sin k\pi t \right) + \frac{A_n}{\sqrt{n}} \cos \frac{n}{2} \pi t \)

\( \Rightarrow X'' = 2c_2 + \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n}{2}-1} \left( -(2k\pi)^2 A_k \cos k\pi t + (2k\pi) B_k \sin k\pi t \right) - \frac{A_n}{\sqrt{n}} \cos \frac{n}{2} \pi t \)

\( \Rightarrow X'' = f(t+1) \)

\( \Rightarrow \begin{pmatrix} 2c_2 = \frac{A_n}{\sqrt{n}} \\ -(2k\pi)^2 A_k = A_k \\ -(2k\pi) B_k = B_k \\ -\frac{A_n}{\sqrt{n}} (\frac{n}{2})^2 = \frac{A_n}{\sqrt{n}} \end{pmatrix} \Rightarrow \begin{pmatrix} c_2 = 0 \\ A_k = \frac{a_k}{-(2k\pi)^2} \\ B_k = \frac{b_k}{-(2k\pi)} \\ A_n = \frac{a_n}{n\pi} \end{pmatrix} \)

\( X(t+1) = x(t) \Rightarrow c_0 \cos (\frac{n}{2} t + \frac{1}{2}) = c_0 \cos (\frac{n}{2} t) \Rightarrow C.C. \)

But the solution is not unique because \( A \)'s, \( B \)'s are unique. \( c_0 \) can be arbitrary.

(c) \( \Rightarrow \) IFT(\( Y \)) will return the inverse discrete Fourier Transform \( X \).

\( \Rightarrow \) Let \( Y = \text{IFT}(X) \), where \( X = (X_0, \ldots, X_{n-1}) = (f_0, \ldots, f_{n-1}) \).

And then \( X(t_j) = X_j \).
(d) if $u \neq 0 \implies c_2 \neq 0$

Note that
\[
\cos(2k\pi(t+1)) = \cos(2k\pi) = 1
\]
\[
\sin(2k\pi(t+1)) = \sin(2k\pi) = 0
\]
\[
\cos((n\pi(t+1)) = \cos(n\pi) \cos \pi t + \cos \pi
\]
\[
\frac{n}{2} - 1 \text{ is a number}
\]

\[
\implies n \text{ is even } \implies \cos(n\pi) = 1
\]

\[
\implies \cos((n\pi(t+1)) = \cos(n\pi) \cos \pi t + \cos \pi
\]

2) $x(t+1) = x(t) = c_2(t+1)^2 - c_3 t^2 - 2c_3 t + 1$

In general, $2c_3 t + 1$ for all $t$

\[
\implies x(t+1) \text{ is not 1-periodic}
\]