DFT 1. a) Let $\omega = e^{-i\frac{2\pi}{4}}$. Plot $\omega^k$, $k = 0, 1, 2, 3, 4$ in the complex plane, explain why $\omega$ is a fourth root of unity and explain why $\omega$ is a primitive fourth root of unity.

b) For general integer $k$, simplify (“telescope”) to show that

$$(1 - \omega^k)(1 + \omega^k + (\omega^k)^2 + (\omega^k)^3) = 0.$$

For what values of $k$ may you conclude that $(1 + \omega^k + (\omega^k)^2 + (\omega^k)^3) = 0$? (Hint: $e^{-i2\pi x} = 1 \Leftrightarrow x$ is an integer).

c) Let

$$F_4 \equiv \frac{1}{\sqrt{4}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix}$$

Show that $F_4^{-1} = F_4$.

d)(438 only) For $x = (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4$, show that the components of $y = F_4x$ obey $y_0 = \overline{y}_0$ and $y_{4-k} = \overline{y}_k$ for $k = 1, 2, 3$.

e)(538 only) Let $y \in \mathbb{C}^4$ be such that the components of $y$ obey $y_0 = \overline{y}_0$ and $y_{4-k} = \overline{y}_k$ for $k = 1, 2, 3$. Show that the components $x_j$, $j = 0, 1, 2, 3$ of $x = F_4^{-1}y$, are all real.

DFT 2. (438 only) Let

$$F_3 \equiv \frac{1}{\sqrt{3}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{bmatrix}$$

where $\omega = e^{-i\frac{2\pi}{3}}$. Assume $F_3^{-1} = F_3$.

a) Suppose $x = (x_0, x_1, x_2)^T \in \mathbb{R}^3$. Let $y = F_3x$, so then $x = \overline{F_3}y$. Show that for each index $j = 0, 1, 2$,

$$x_j = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} e^{i\frac{2\pi k j}{3}} y_k.$$

b) Use the result from a) to show that the function $Q$ defined by

$$Q(t) = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} e^{i2\pi kt} y_k$$
interpolates the data \((t_j, x_j), j = 0, 1, 2\), that is, \(Q(t_j) = x_j\) for \(j = 0, 1, 2\).

c) Write \(Q(t) = Q_0 + Q_1(t) + Q_2(t)\) where
\[
Q_0 \equiv \frac{1}{\sqrt{3}} y_0, \quad Q_1 \equiv \frac{1}{\sqrt{3}} e^{i2\pi t} y_1, \quad Q_2(t) \equiv \frac{1}{\sqrt{3}} e^{i4\pi t} y_2.
\]

Show that \(Q_2(t_j) = \overline{Q_1(t_j)}\) for each \(j = 0, 1, 2\).

d) Define \(P_3(t) \equiv Q_0 + Q_1(t) + \overline{Q_1(t)}\).

Show that \(P_3(t)\) interpolates the data \((t_j, x_j), j = 0, 1, 2\). (Hint: \(P_3(t) - \overline{Q_1(t)} + Q_2(t) = Q(t)\).)

\textit{DFT}_3. (538 only) Let \(n\) be a general positive integer. Let \(\omega = e^{-i\frac{2\pi}{n}}\), and let \(F_n\) be the Fourier matrix. Then with 0-base indexing, \((F_n)_{j,k} = \frac{1}{\sqrt{n}} \omega^{jk}\). Assume \(F_n^{-1} = F_n^\top\).

a) Suppose \(x \in \mathbb{R}^n\). Let \(y = F_n x\), so then \(x = F_n^\top y\). Show that for each index \(j = 0, \ldots, n-1\),
\[
x_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{i\frac{2\pi k j}{n}} y_k.
\]

b) Use the result from a) to show that the function \(Q\) defined by
\[
Q(t) \equiv \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{i2\pi k t} y_k
\]
interpolates the data \((t_j, x_j), j = 0, \ldots, n - 1\).

c) Assume that \(n\) is odd and \(n \geq 3\). Write \(Q(t) = Q_0 + Q_1(t) + Q_2(t)\) where
\[
Q_0 \equiv \frac{1}{\sqrt{n}} y_0, \quad Q_1 \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} e^{i2\pi k t} y_k, \quad Q_2(t) \equiv \frac{1}{\sqrt{n}} \sum_{k=n+1}^{n-1} e^{i2\pi k t} y_k.
\]

Show that \(Q_2(t_j) = \overline{Q_1(t_j)}\) for each \(j = 0, \ldots, n - 1\). You may assume that \(y_{n-k} = \overline{y_k}\), \(k = 1, \ldots, n - 1\).

d) Define \(P_n(t) \equiv Q_0 + Q_1(t) + \overline{Q_1(t)}\).

Show that \(P_n(t)\) interpolates the data \((t_j, x_j), j = 0, \ldots, n - 1\). (Hint: \(P_n(t) - \overline{Q_1(t)} + Q_2(t) = Q(t)\).)

\textit{DCT}_1. Let \(a, b, c\) be positive constants and let
\[
C = a \begin{bmatrix} a & a & a & a \\ b & c & -c & -b \\ a & -a & -a & a \\ c & -b & b & -c \end{bmatrix}
\]
a) Compute $CC^T$ for general $a, b, c$. Then show that in the case $a = \cos(\pi/4) = \frac{1}{\sqrt{2}}$, $b = \cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$, $c = \cos(3\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2}$, it follows that $CC^T = I$, the identity matrix.

b) From $CC^T = I$ it then follows that $C^T C = I$. Explain why.

c) Suppose we interpolate data $(t_j, x_j) = (j, x_j)$, $j = 0, 1, 2, 3$ by means of a linear combination $\sum_{k=0}^{3} y_k f_k(t)$ where the functions $f_k(t)$ are given and the coefficients $y_k$, $k = 0, 1, 2, 3$ are to be determined. Write out four individual linear equations for the unknowns $y_k$. Then rewrite the system as a matrix system, using

$$A = \begin{bmatrix} f_0(t_0) & f_1(t_0) & f_2(t_0) & f_3(t_0) \\ f_0(t_1) & f_1(t_1) & f_2(t_1) & f_3(t_1) \\ f_0(t_2) & f_1(t_2) & f_2(t_2) & f_3(t_2) \\ f_0(t_3) & f_1(t_3) & f_2(t_3) & f_3(t_3) \end{bmatrix}$$

to denote the coefficient matrix and $y = (y_0, y_1, y_2, y_3)^T$ to denote the unknown vector.

d) Assume the functions $f_k$ are chosen such that that $A = C^T$. Find a formula for the solution $y = (y_0, y_1, y_2, y_3)^T$ of the system in part c). What is the name of this formula?

e) (538 only) Show that for the choice of functions $f_0(t) = \frac{1}{2}$, $f_k(t) = \frac{1}{\sqrt{2}} \cos \frac{k(2t+1)\pi}{8}$, $k = 0, 1, 2$, the matrix $A$ in part c) is given by the formula $A = C^T$.

$DCT_2$. a) Verify that $C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is an orthogonal matrix.

b) Compute the two-dimensional discrete cosine transform $Y = CXC^T$ of the data

$$X = \begin{bmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

c) Let $\hat{Y}$ denote the result of applying a low-pass filter to $Y$, such that $\hat{Y}_{i,j} = Y_{i,j}$ for $(i,j) \neq (1,1)$ and $\hat{Y}_{1,1} = 0$. Calculate $\hat{X}$, the corresponding “reconstructed” version of $X$. Compare $\hat{X}$ to $X$: that is, how do the elements of $\hat{X}$ differ from those of $X$?

$EV_1$. Let

$$A = \begin{bmatrix} 1 & \frac{1}{4} \\ 1 & 1 \end{bmatrix}$$

and let $\lambda_1$ and $\lambda_2$ denote the eigenvalues of $A$, arranged so that $|\lambda_1| > |\lambda_2|$.

a) Calculate $\lambda_1$, $\lambda_2$ and the corresponding eigenvectors $v_1$ and $v_2$.

b) Define $x^{(0)} = (1, 1)^T$. Find constants $c_1$, $c_2$ such that $x^{(0)} = c_1 v_1 + c_2 v_2$.

c) Perform two steps of power iteration starting with $x^{(0)}$. Estimate the dominant eigenvalue of $A$ by the Rayleigh quotient $\frac{x^T Ax}{x^T x}$ with $x = x^{(0)}$, $x = x^{(1)}$, $x = x^{(2)}$. 


d) Find the general formula for $x^{(k)}$ in terms of $c_1, c_2, v_1$ and $v_2$.

e) (538 only) Perform two steps of inverse power iteration starting with $w^{(0)} = (1, 1)^T$ (shift $s = 0$.) Then $w^{(k+1)} = A^{-1}w^{(k)}$, $k = 0, 1$. Estimate the dominant eigenvalue of $A^{-1}$ by the Rayleigh quotient $\frac{w^TA^{-1}w}{w^Tw}$ with $w = w^{(0)}$, $w = w^{(1)}$, $w = w^{(2)}$.

f) (538 only) Show that $\lim_{k \to \infty} \frac{(w^{(k)})^T A^{-1} w^{(k)}}{(w^{(k)})^T w} = \frac{1}{\lambda_2}$.

EV2. The “second eigenvalue of the Google matrix” theorem of Haveliwala and Kamvar states:

Let $q$ be a real number, $0 \leq q \leq 1$.
Let $P$ be an $n \times n$ matrix such that $P^T$ is stochastic.
Let $E$ be the $n \times n$ matrix $E = ev^T$, where $e$ is the $n$-vector whose elements are all 1, and $v$ is an $n$-vector such that $v_i \geq 0$ for each $i$ and $v_1 + v_2 + .. + v_n = 1$.
Define the matrix $G = [qE + (1 - q)P]^T$.
Then the second eigenvalue of $G$ obeys $|\lambda_2| \leq 1 - q$. (The first eigenvalue is $\lambda_1 = 1$ and the eigenvalues are ordered so that $|\lambda_1| \geq |\lambda_2| \geq .. \geq |\lambda_n|$).

a) Let

$$
A = \begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}, \quad \text{then let } P = \begin{bmatrix}
0 & 0 & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
$$

be obtained from $A$ by dividing each row of $A$ by the sum of the elements in that row.
Show that $P^T$ is a stochastic matrix.

b) Show that the $3 \times 3$ matrix

$$
E = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

can be expressed as $ev^T$, where

$$
e = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad v^T = \begin{bmatrix}
v_1 & v_2 & v_3
\end{bmatrix},
$$

with each $v_i \geq 0$ and $v_1 + v_2 + v_3 = 1$ (find $v$).

c) Define the matrix $G = \left[\frac{1}{3}E + \frac{3}{4}P\right]^T$. Apply the “second eigenvalue” theorem to bound the second eigenvalue of this $3 \times 3$ matrix $G$.

d) Let $p = (p_0, p_1, p_2)^T$ denote the eigenvector of $G$ for eigenvalue $\lambda = 1$, normalized such that $p_0 + p_1 + p_2 = 1$. Which component of $p$ would you expect to have the largest value?