Math 438/538 Test 1 Feb. 23 2012

Each student may bring in a calculator. PROHIBITED: calculators with built-in computer algebra systems, handhold, tablet, or laptop computers, including PDAs, electronic writing pads or devices with pen-input, calculators built into cell phones or any other electronic communication devices. Students may bring in one 4x6 index card or two 3x5 index cards, written both sides, with any information. If you need a formula, ask and it may be displayed. Please try all questions.

O1. Consider 
\[ f(x) = \frac{x^3}{3} + \frac{x^2}{2} - x + 1 \quad \text{for} \quad x \geq 0. \]

a) Show that \( f''(x) > 0 \) on \([0, \infty)\). Use this result to show that \( f'(x) \) has the property, 
\[ x^2 > x_1 > x_2 \Rightarrow f'(x_2) > f'(x_1), \]
and so \( f'(x) \) is strictly increasing on \([0, \infty)\).

b) Show \( f'(0) < 0 \) and \( f'(1) > 0 \). Then explain why there must be some point \( c \), \( 0 < c < 1 \)
at which \( f'(c) = 0 \).

c) Show further that \( f'(x) < 0 \) for \( 0 < x < c \) and \( f'(x) > 0 \) for \( x > c \). Conclude that \( f \) is unimodal on \([0, 1]\).

d) Without computing any iterates, how many steps of Golden Section Search are needed so that the error in the approximation to \( x^* = c \), is guaranteed less than 0.0005? Note that \( g = \frac{\sqrt{5} - 1}{2} \approx 0.618 \), approximately, and that the approximation to \( x^* \) is the center of the current interval \([a_k, b_k]\).

e) Perform one step of G.S.S, using the value \( g = 0.618 \).

f) (538 only) A computer program written to perform this G.S.S. computation using a very accurate value for \( g \), shows that G.S.S. chooses the right subinterval, then the left, then the right, then the left and so on. Explain why this happens.

O2. Let 
\[ f = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 - x_1 + x_2 \quad (1) \]

a) Calculate the gradient vector and Hessian matrix 
\[ \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \]

c) Show that \( -\nabla f = (1, -1)^T - Ax \). (hence the name “residual”)

d) The Conjugate Gradient Search method is defined as follows:

Given initial guess \( x_0 \), let \( d_0 = r_0 = -\nabla f(x_0) \).

for \( i = 1, 2, \ldots \) : Let \( \alpha = \alpha_i \) minimize \( f(x_{i-1} + \alpha d_{i-1}) \),
\[ x_i = x_{i-1} + \alpha_i d_{i-1}, \]
\[ r_i = -\nabla f(x_i), \]
\[ \beta_i = \frac{r_i^T r_i}{r_{i-1}^T r_{i-1}}, \text{ and} \]
\[ d_i = r_i + \beta_i d_{i-1} \]

Carry out one iteration of C.G.S., starting with \( x_0 = (0, 0)^T \).
e) Show that the search directions \( d_0 \) and \( d_1 \) obey \( d_0^T A d_1 = 0 \) (are “conjugate”)

**O3.** In each of the following, \( f \) is the objective function, and the purpose of the method is to minimize \( f \).

True or False:
a) To be certain that Golden Section Search will converge, \( f \) must be unimodal on the interval \([a, b]\) of interest.
b) If Successive Parabolic Interpolation is applied to a quadratic function, the iteration will converge.
c) For Newton’s Method to apply, \( f \) must be twice differentiable.
d) If Newton’s Method is applied to a function of the form \( f = x^T A x - b \) where \( A \) is a symmetric, positive definite matrix, then the iteration will converge.
e) For the Nelder-Mead Method to apply, \( f \) must be differentiable.
f) If the Conjugate-Gradient-Search Method converges to a point \( x_* \), then \( f \) has a local minimum value at \( x_* \).
g) Derivative-free methods are more accurate that methods that require derivatives.
h) If \( f(x, y) \) obeys \( f(y, x) = f(x, y) \) and \( f \) has a local minimum at \((a, b)\), then \( f \) also has a local minimum at \((b, a)\).

**R1.** Consider the Linear Congruential Generator (LCG) given by

\[ v_{n+1} = 5v_n + 1 \mod 16, \quad w_n = \frac{v_n}{16} \]

for \( n = 0, 1, 2, \ldots \) with seed \( v_0 = 1 \).

a) Calculate the terms of the sequence \( \{v_n\} \) until it begins to repeat. What is the period \( p \)?
b) Calculate \( \bar{v} = E[\{v_n\}] \), the mean value (based on one period), and then calculate \( \bar{w} = E[\{w_n\}] \) by using the formula \( \bar{w} = \frac{1}{16} \bar{v} \).
c) Calculate \( \bar{X} = E[X] \) for random numbers \( X \) uniformly distributed on \([0, 1]\): recall the probability density function is \( p(x) = 1 \). Compare \( \bar{X} \) to \( \bar{w} \) from part b).
d) For the sequence \( v_n \), calculate \( var(\{v_n\}) = E[\{(v_n - \bar{v})^2\}] \) (based on one period). Calculate \( var(\{w_n\}) = E[\{(w_n - \bar{w})^2\}] \) by using the formula \( var(\{w_n\}) = var(\{\frac{1}{16} v_n\}) = \frac{1}{256} var(\{v_n\}) \).
e) Calculate \( var(X) = E[(X - \bar{X})^2] \) for random numbers \( X \) uniformly distributed on \([0, 1]\). Compare \( var(X) \) to \( var(\{w_n\}) \) from part d).
\( R_2 \). Let \( f(x) = x^{1/2} \). Suppose \( X \) is a random variable uniformly distributed on \([0,1]\), that is, with probability density \( p(x) = 1 \).

a) Calculate \( E[f(X)] = \int_0^1 f(x)p(x)dx \).

b) Calculate \( \text{var}(f(X)) = E[(f(X) - \bar{f})^2] \), where \( \bar{f} = E[f(X)] \).

c) Use one period of the sequence \( w_n \) described in problem \( R_1 \) above, to approximate \( \int_0^1 f(x)dx \).

d) Show that for any positive integer \( M=1,2,3,\ldots, \)
\[
\frac{1}{Mp} \sum_{n=1}^{Mp} f(w_n) = \frac{1}{p} \sum_{n=1}^{p} f(w_n).
\]

e) (538 only) Show that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(w_n) = \frac{1}{p} \sum_{n=1}^{p} f(w_n).
\]

Hint: Write \( N = Mp + p', \) where \( 0 \leq p' < p \). Then as \( N \to \infty, \) also \( M \to \infty \) and \( \frac{1}{N} = \frac{1}{Mp(1+\epsilon)} = \frac{1}{Mp} \left( 1 - \epsilon + O(\epsilon^2) \right) \) where \( \epsilon = \frac{p'}{Mp} \to 0 \).

\( R_3. \) a) Apply Ito’s lemma to show that \( y = \sigma B_t + (r - \sigma^2/2)t \) satisfies the stochastic differential equation (S.D.E.)
\[
dy = (r - \sigma^2/2)dt + \sigma dB_t.
\]

b) Apply Ito’s lemma to show that \( S = S_0 e^{\sigma B_t +(r-\sigma^2/2)t} \) satisfies the S.D.E.
\[
dS = rS dt + \sigma S dB_t
\]

c) (538 only) A Wiki article gives the following form of Ito’s lemma: If \( dx = a \, dt + b dB_t \) and \( y = g(t,x) \), then
\[
dy = \left( a \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 g}{\partial x^2} \right) dt + b \frac{\partial g}{\partial x} dB_t.
\]

The version of Ito’s lemma which appears in the textbook is:
If \( y = f(t,x) \), then
\[
dy = \frac{\partial f}{\partial t}(t,x) dt + \frac{\partial f}{\partial x}(t,x) dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t,x) dx \, dx.
\]

where the \( dx \, dx \) term is interpreted by using the identities \( dt \, dt = 0, \, dt \, dB_t = 0, \, dB_t \, dt = 0 \) and \( dB_t \, dB_t = dt \).

Show that the version in the Wiki article follows from the version in the textbook. Assume \( a \) and \( b \) are constants.