9.1 Random numbers
A.9 Random Numbers.

Consider random numbers, uniform on \([0, 1]\).

Each number drawn, \(X_i\) with number drawn.

Each bucket (bin) is equally probable:

\[
P(0.3 < X_i < 0.4) = (0.4 - 0.3) = 0.1
\]

Generally:

\[
P(a < X_i < b) = b - a.
\]

The uniform distribution has probability density function

\[
A(x) = 1
\]

Draw very large number of \(X_i\)'s: on average, find bins equally filled.
The density function \( f(x) \) is such that

\[
P(x < c) = \int_{\infty}^{c} f(x) \, dx
\]

\[
P(x \geq c) = 1 - P(x < c)
\]

Since \( P(x < 1) = 1 \), \( \int_{\infty}^{1} f(x) \, dx = 1 \).

Linear Congruential Generators

\( a, c, c, d \)

are algorithms to generate "pseudo" random numbers, of the form,

\[
\begin{align*}
    x_i &= (ax_{i-1} + b) \mod m \\
    u_i &= x_i / m
\end{align*}
\]
\[ a = 7^5 \quad b = 0 \quad m = 2^{31} - 1. \]

gives LCG called the Minimal Standard Random Number Generator.

Claim: the sequence \( \{X_i\} \) only repeats after \( m-1 \) draws.

(period to \( 2^{31}-2 \), the maximum possible).

For most purposes, pseudo-random numbers are good enough.
Suppose $x_i$ is random variable with prob. density function $f(x)$

\[ P(x_i < c) = \int_{s=a}^{s=c} f(s) \, ds \]

How do we compute $F(x_i)$?

Suppose we can generate $\{x_i\}$ uniformly distributed on $[0, 1]$.
Claim: there is a function $G(u)$ such that $X_i = G(U_i)$ has the desired prob. density $f$. That is, $P(X_i < c) = \int_{s=a}^{c} f(s) \, ds$.
\[
\begin{align*}
\kappa_i &< c \\
 \iff F(\kappa_i^c) &< F(c) \\
 u_i &< c' \\
u_i &= G(u_i) \\
P(\kappa_i < c) &= P(F(\kappa_i^c) < F(c)) \\
&= P(u_i < c').
\end{align*}
\]

But, \( u_i \) is uniform, so \( P(u_i < c') = \int_0^{c'} 1 \, du = c' \).

\[
P(\kappa_i < c) = c' = F(c) = \int_a^c f(s) \, ds.
\]

\[ \exists \kappa_i \text{ given by } \kappa_i = G(u_i), \]

has desired prob. density function.
For the “normal” (Gaussian) distribution with mean \( \mu = 0 \) and standard deviation \( \sigma = 1 \), the probability density function is

\[
\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \, ds = 1.
\]

\[
P(x_1 < c) = \int_{-\infty}^{c} f(x) \, dx
\]

\[
P(x_1 < \infty) = \int_{-\infty}^{\infty} f(x) \, dx = 1.
\]
Efficient algorithms for generating pseudo-random numbers with “normal” distribution, include

Box-Muller #1

which uses pairs $u_1, u_2$ of uniformly distributed pseudo-random numbers, to produce pairs $x_1, x_2$ of “normal” distributed pseudo-random numbers, saves evaluating $G$.

Box Muller # 2

saves sine & cosine evaluations as compared with Box Muller #1.