8.3.1 Finite difference method for elliptic equations
Finite Differences: Elliptic equations

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad u = g \text{ on } \partial \Omega \]

in some region \( \Omega \)

\[ \Omega = \{ x_{e} < x < x_{l}, \quad y_{b} < y < y_{t} \} \]

left \[ \quad \text{top} \]

right \[ \quad \text{bottom} \]

\( \hat{\Omega} \) \( f(x, y) \equiv 0 \), Poisson's eqn.

\( \hat{\Omega} \) \( f(x, y) \equiv 0 \), Laplace's eqn.

\( y_{N} = y_{N+1} = y_{t} \)

\[ k = \frac{y_{t} - y_{b}}{N} \]

\( y_{0} = y_{b} \)

\( k_{e} = k_{m+1} \)

\( x_{e} = x_{b} \quad x_{l} \quad x_{k} \quad x_{l} \quad x_{m} \)

\( x_{m} = x_{M} \)

\[ h = \frac{x_{l} - x_{e}}{M} \]
Approximate Laplace's Eqn by at \((x_i, y_j)\) by:

\[
\frac{w_{i+1,j} + w_{i-1,j} - 2w_{i,j}}{h^2} + \frac{w_{i,j+1} + w_{i,j-1} - 2w_{i,j}}{k^2} = 0
\]

for \(i = 1, 2, \ldots, m,\)

\(j = 1, 2, \ldots, n.\)

These involve \(w_{0,j}, \ w_{i+1,j}, \ w_{i,j+1};\)

\(w_{i,j} = g_1(x_i)\) bottom data

\(w_{i,n+1} = g_2(x_i)\) top data

\(w_{0,j} = g_3(y_j)\) left data

\(w_{m+1,j} = g_4(y_j)\) right data
Adopt a numbering scheme for the unknowns:

\[ X \]

\[ Y \]

\[ U_{1} \quad U_{2} \quad U_{3} \]

\[ V_{1} \quad V_{2} \quad V_{3} \quad V_{4} \]

\[ V_{4M+1} \quad V_{4M+2} \quad V_{4M+3} \quad V_{4M+4} \]

\[ N_{1} \quad N_{2} \quad N_{3} \quad N_{4} \]

\[ N_{4M+1} \quad N_{4M+2} \quad N_{4M+3} \quad N_{4M+4} \]
\[
\frac{w_{i,j-1}}{h^2} + \frac{w_{i-1,j}}{h^2} + \left[(-\frac{2}{h^2}) + (-\frac{2}{k^2})\right] w_{i,j} + \frac{w_{i,j+1}}{h^2} + \frac{w_{j+1}}{k^2} = 0
\]

- \(w_{i,j+1}\) appears after, by \(M+1\).
- \(w_{i,j}\) appears before in \(i\) \(\text{th index}\).
- \(w_{i,j-1}\) appears before in array \(N\); \(M+1\) indices

Find large matrix \(A\):
- some diagonal entries \(1\) (all other entries in row 0)
- some diagonal entries \(-\frac{2}{h^2} - \frac{2}{k^2}\)
- entry \(\frac{1}{h^2}, 0, 0, \ldots, \frac{1}{h^2}, -\frac{2}{h^2} - \frac{2}{k^2}, \frac{1}{h^2}, 0, 0, \ldots, \frac{1}{k^2}, 0, \ldots\)
Ex by hand.

Note: not the same \( v_1, v_2, \ldots \)

\[ \nabla^2 u = 0 \]

\[ \left\{ \begin{array}{l} k = k \\ \text{cancel } \frac{1}{h^2} \end{array} \right. \]

\[ a \, w_{i+1, j} + w_{i-1, j} + w_{i, j+1} + w_{i, j-1} - 4 \, w_{i, j} = 0 \]

Eqn 1:

Eqn 2:

Eqn 3:

\[ \text{etc.} \]

\[ \text{Eqn 9: in all: 9 eqns.} \]