13.1 Unconstrained optimization without derivatives
Ch. 13 Optimization

Concentrate on minimization: maximization basically the same.
(just change the objective function)

The function \( f = f(x) \) we try to minimize, is called the objective function.

Optimization problems are classified as unconstrained, or constrained:

ex: minimize \( f(x, y) \) given by (complicated formula)

minimize \( f(x, y) \) given by (complicated formula)

subject to: \( y < x^2 \)
nonlinear constraint
Linear programming deals with optimization of linear functions, subject to linear constraints: special topic.

Generally: unconstrained minimization easier than constrained, linear objective functions easier than nonlinear.

One-dimensional problems easier than two, two easier than three, etc.

Problems with a function to minimize, where derivatives can be found, are easier than problems where derivatives are not available.
Minimization of functions $f(x)$ of 1 variable $x$

Definition: $f$ is unimodal if:

- There is one point at which $f$ attains its minimum value; say $x = x_0$,

and:

- $f$ is increasing for $x > x_0$

- decreasing for $x < x_0$. 

Example: $f(x) = |x+1|$. 

\[ f(x) = |x + 1| \]
"Golden section" and the Golden Section Search.

\[ x_1 = 1 - g \quad x_2 = g \]

Interval scaled by \( g \)

\[ 0 \quad g x_1 \quad g x_2 \quad g \]

\( g \) has property:

\[ g x_2 = x_1 = 1 - g \]

\[ g^2 = 1 - g \]

\[ g^2 + g - 1 = 0 \]

\[ g = \frac{(-1 \pm \sqrt{1 + 4})}{2} \]

Since \( g > 0 \), choose '+':

\[ g = \frac{(-1 + \sqrt{5})}{2} \]
C.S.S.: Suppose $f$ unimodal (min.) on $[a, b]$, say $[0, 1]$

\[\begin{array}{cccc}
& 0 & \gamma & x_2 & 1 \\
\end{array}\]

From values $f(x_1), f(x_2)$ obtain information:

* If $f(x_2) > f(x_1)$, then it must be that $x_1 < x_2$. (Case cannot be located in $x_1 \geq x_2$)

* If $f(x_2) < f(x_1)$, it must be that $x_1 > x_2$. (Sharp case $f(x_2) < f(x_1)$).
then

\[ \frac{x_i}{x_1} = \frac{x}{x_1} \]

now need \( f(x_1') \) (one for evaluation)

(already know \( f(x_2') = f(x_1) \) previous step)

then either

\[ x_2' = x_1' \]

or

\[ x_2' = x_2'' \]

now need \( f(x_2') \) (just one for eval; know \( f(x_1') = f(x_2) \) previous step)
Successive Parabolic Interpolation:

Similar, but:

fit a parabola to \( f \), based on values at 3 points.

\[ \begin{align*}
    y_1 &= f(x_1) \\
    y_2 &= f(x_2) \\
    y_3 &= f(x_3)
\end{align*} \]

Form \( P(x) = a + b x + c x^2 \) such that

\[ \begin{cases}
    P(x_1) = y_1 \\
    P(x_2) = y_2 \\
    P(x_3) = y_3
\end{cases} \]

then solve \( P'(x) = 0 \) for \( x_4 \);

evaluate \( f(x_4) = y_4 \).

New \( (x_1', y_1'), (x_2', y_2), (x_3', y_3') \): discard previous pair \( x_i, y_i \)

when \( y_i = \max(y_1, y_2, y_3, y_4) \).
Related to unconstrained minimization

\[ y = c_1 + c_2 t \]

\((t_1, y_1), (t_2, y_2), (t_3, y_3)\) are given. The line \( y = c_1 + c_2 t \) approximates this data.

\[
\text{err}_1 = y_1 - (c_1 + c_2 t_1), \quad \text{err}_2 = y_2 - (c_1 + c_2 t_2), \quad \text{err}_3 = y_3 - (c_1 + c_2 t_3)
\]

To minimize \( \beta(c_1, c_2) = \sum_{i=1}^{3} (\text{err}_i)^2 \), by choosing \( c_1, c_2 \) so \( \beta \) attains its minimum value.

One way to regard this problem: “solve” overdetermined system in sense of finding a least squares solution:

\[
\begin{bmatrix}
1 & t_1 \\
1 & t_2 \\
1 & t_3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} - 
\begin{bmatrix}
\text{err}_1 \\
\text{err}_2 \\
\text{err}_3
\end{bmatrix}
\]

\[ M \tilde{c} = \tilde{y} - \tilde{e} \]
The vector $\hat{c}$ (least squares coefficients) satisfies

$$M^T M \hat{c} = M^T \hat{y}$$

of form, $A \hat{c} = \hat{b}$

where: $A = M^T M$ is symmetric,

and: $\hat{x}^T A \hat{x} > 0$ any $\hat{x} \neq (0)$.

$A$ is positive definite.