NAME (please print legibly): __________________________________________
Your University ID Number: ________________________________________

- Please show all your work. You may use back pages if necessary.

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<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
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1. (10 points) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $0 < x < 3$, $0 < y < 1$ with boundary conditions $u(0, y) = u(3, y) = 0$ for $0 < y < 1$, $u(x, 0) = 0$ and $u(x, 1) = \sin \left( \frac{\pi x}{3} \right)$ for $0 < x < 3$. 
2. (10 points) The temperature in a long cable is modelled by

\[
\frac{\partial v}{\partial t} = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} \right] \text{ for } 1 < r < 2, \ 0 < \theta < \pi, \ -\infty < z < \infty, \ t > 0.
\]

The horizontal coordinate is \(z\). The inner region \(0 \leq r \leq 1\) is maintained at a constant temperature.

a) Verify that each term of \(v = a_0 + b_0 \ln(r) + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos(n\theta) + (c_n r^n + d_n r^{-n}) \sin(n\theta)\) satisfies the PDE and is \(2\pi\)-periodic in \(\theta\). The \(a_n, b_n, c_n\) and \(d_n\) are arbitrary constants.

b) Cooling and heating maintains temperatures \(v(1, \theta) = 0\) and \(v(2, \theta) = 15 - 15 \sin 2\theta\), \(0 < \theta < \pi\). Solve the boundary value problem consisting of the PDE and these boundary conditions.
3. **(10 points)** The motion of a circular drum head of radius 1 foot is studied by examining solutions of wave equation of the form \( u(r, t) = f(r)h(t) \). Let \( c \) be the wave-speed in ft/sec. Then \( h \) and \( f \) satisfy

\[
\begin{align*}
  h''(t) + c^2 \lambda h(t) &= 0, \\
  \frac{d}{dr} \left( r \frac{df(r)}{dr} \right) + \lambda rf(r) &= 0
\end{align*}
\]

where \( \lambda \) is the (positive) separation constant. For particular values of \( \lambda \), the eigenvalues, there are non-trivial solutions \( f \) such that \(|f(0^+)| < \infty \) and \( f(1) = 0 \).

a. Let \( \phi(r) = 1 - r^2 \). Use this trial function in a Rayleigh quotient, to approximate the first eigenvalue \( \lambda_1 \).

b. The drum head is set in motion with initial data \( u_0(r, 0) = f_1(r) \), where \( f_1(r) \) is the eigenfunction corresponding to \( \lambda_1 \). The subsequent motion has frequency 48 Hertz. What is the wavespeed \( c \)? In your calculation use the approximate value for \( \lambda_1 \) from a).
4. **(10 points)** a) Find the Greens’s function \( G(x, x_0) \) on \([0, \infty)\) such that

\[
\frac{d^2 G}{dx^2} - \frac{dG}{dx} = \delta(x - x_0), \quad G(0, x_0) = 0, \quad |G| \text{ bounded}
\]

b) What boundary value problem does the function \( u(x) = \int_{x=0}^{\infty} G(x, \bar{x}) f(\bar{x}) d\bar{x} \) satisfy? (You may differentiate under the integral sign as needed, without justification)

c)(518 only) Suppose \( \lim_{x \to \infty} f(x) = \alpha \), a constant. For what \( \alpha \) is \( u \) defined?
5. (10 points) a) Solve the heat equation \( u_t = k(u_{xx}) \) for \( t > 0 \) on \( 0 < x < \infty \) with initial conditions \( u(x, 0) = e^{-x^2} \), by means of Fourier Cosine transforms.

Table of Fourier Cosine Transforms: 
\[
C\{f(x, t)\} = \frac{2}{\pi} \int_0^\infty f(x, t) \cos \omega x dx = F(\omega, t),
\]
\[
C\left\{ \frac{\partial f}{\partial t}(x, t) \right\} = \frac{\partial F}{\partial t}(\omega, t), \quad C\left\{ \frac{\partial^2 f}{\partial x^2}(x, t) \right\} = \frac{2}{\pi} \frac{\partial f}{\partial x}(0, t) - (\omega)^2 F(\omega, t),
\]
\[
C\{e^{-\alpha x^2}\} = \frac{1}{\sqrt{\pi \alpha}} e^{-\omega^2/(4\alpha)} \quad C^{-1}\{e^{-\beta \omega^2}\} = \sqrt{\frac{\pi}{4\beta}} e^{-x^2/(4\beta)}
\]

b. (518 only) Explain what happens to the solution from part b for negative values of \( t \).
6. (10 points) Consider $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, \quad 0 < t < \infty, \quad -\infty < x < \infty

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 \leq x \leq 4$$

$$u(0, t) = u(4, t) = 0, \quad t \geq 0$$

a. Suppose $c = 1$. On the grid given on the next page, sketch both characteristics through the point $(x, t) = (2, 0)$. Sketch both characteristics through the point $(x, t) = (3, 0)$. Also sketch all reflected characteristics.

b. Suppose $u(x, 0) = f(x) = \begin{cases} 
0 & \text{for } 0 \leq x < 2 \\
0.2 & \text{for } 2 \leq x < 3 \\
0 & \text{for } 3 \leq x \leq 4 
\end{cases}$

Sketch this initial data along the $x$ axis of the grid on the next page. Sketch the odd extension of this data along the $x$ axis for $-4 \leq x \leq 0$. Sketch the 8-periodic extension of the odd extension along the $x$ axis for $4 \leq x \leq 8$.

c. The characteristics and reflected characteristics in a) serve as boundaries of subregions where the solution $u(x, t)$ for initial data in c), is constant. Sketch these subregions on the grid, and label each subregion with the constant value of the solution $u$ on that subregion. Hint: Use the D'Alembert solution with 8-periodic initial data.