These notes provide roughly 5 class hours worth of supplemental material for Math 306, our first course in ordinary differential equations.

The intended audience consists of students in Engineering, the Sciences, and Mathematics who have taken the Calculus sequence and have seen continuous functions, differentiable functions, sequences, series, and power series for functions. It is not assumed that the students have previously seen the notion of a function analytic at a point, or the technique of change-of-index in a power series.

After covering the material in these notes and working through the exercises and the associated Maple workspace, a student should be able to find the general solution in terms of power series of DEs such as Airy’s equation, and should be able to apply the method of Frobenius to find at least one solution of DEs with regular singular points such as Bessel’s equation.

Working through the Maple workspace familiarizes the student with (i) the language of power series (ii) change of index, (iii) the idea of splitting off terms of a power series in order to add power series, (iv) power series solution of Airy’s equation including plots of approximate solutions and of the exact solutions, (v) solution of Bessel’s equation by the method of Frobenius.

In these notes, we adopt the convention that series with $t$ as the variable will be developed about the origin, i.e. $t = 0$. (The change of variables $x - x_0 = t$ may be used if the original problem is given at a point $x = x_0$ other than the origin).

1 Introductory examples

These first two examples show how using power series to solve a DE, can be thought of as a version of the method of undetermined coefficients.

1.1 $\frac{dy}{dt} + y = 1 + t + \frac{1}{2}t^2$

To find a solution $y$ of this equation, we apply the method of undetermined coefficients (a “lucky guess” method). We look at the form of the function $1 + t + \frac{1}{2}t^2$. Since this is a polynomial of degree 2, we try to find a solution $y$ of
the form
\[ y = a_0 + a_1 t + a_2 t^2 \]
where \( a_0, a_1 \) and \( a_2 \), the coefficients of the powers of \( t \) in \( y \), are to be determined by substitution of \( y \) into the differential equation. We find
\[
\frac{dy}{dt} = a_1 + 2a_2 t, \quad \text{so}
\]
\[
\frac{dy}{dt} + y = a_1 + a_0 + (2a_2 + a_1) t + a_2 t^2.
\]
Recall that we want \( y \) to satisfy
\[
\frac{dy}{dt} + y = 1 + t + \frac{1}{2} t^2.
\]
If we determine the coefficients \( a_0, a_1 \) and \( a_2 \) so that
\[ a_1 + a_0 + (2a_2 + a_1) t + a_2 t^2 = 1 + t + \frac{1}{2} t^2 \]
for all \( t \), then \( y \) will be a solution of the differential equation. Therefore if \( a_0, a_1 \) and \( a_2 \) satisfy
\[
a_1 + a_0 = 1 \quad 2a_2 + a_1 = 1 \quad a_2 = \frac{1}{2}
\]
then \( y \) will satisfy the DE. We find \( a_0 = 1, a_1 = 0, a_2 = \frac{1}{2} \) and so
\[
y = a_0 + a_1 t + a_2 t^2 = 1 + \frac{1}{2} t^2.
\]

1.2 \( \frac{dy}{dt} + y = 1 + t + \frac{1}{2} t^2 + \ldots + \frac{1}{n} t^n + \ldots \)

The function \( 1 + t + \frac{1}{2} t^2 + \ldots + \frac{1}{n} t^n + \ldots \) is a power series, so we look for a solution \( y \) of the form
\[
y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + a_{n+1} t^{n+1} + \ldots
\]
where \( a_0, a_1, a_2 \ldots \) are the coefficients to be determined by substitution of \( y \) into the differential equation.

1.2.1 Formal construction

We differentiate the series \( y \) term by term, obtaining a series for \( \frac{dy}{dt} \). Then we add series, term by term:
\[
\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + na_n t^{n-1} + (n + 1)a_{n+1} t^n + \ldots, \quad \text{so}
\]
\[ \frac{dy}{dt} + y = a_1 + a_0 + (2a_2 + a_1)t + (3a_3 + a_2)t^2 + \ldots + ((n + 1)a_{n+1} + a_n)t^n + \ldots \]

Recall that we want \( y \) to satisfy

\[ \frac{dy}{dt} + y = 1 + t + \frac{1}{2}t^2 + \ldots + \frac{1}{n!}t^n + \ldots \]

If we determine the coefficients \( a_0, a_1, a_2 \ldots \) so that

\[ a_1 + a_0 + (2a_2 + a_1)t + (3a_3 + a_2)t^2 + \ldots + ((n + 1)a_{n+1} + a_n)t^n + \ldots \]

\[ = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \ldots + \frac{1}{n!}t^n + \ldots \]

for all \( t \), then \( y \) will be a solution of the differential equation. By equating the powers of \( t \) on each side, we see that if the coefficients \( a_n, n=0,1,2,\ldots \) satisfy the algebraic equations

\[ a_1 + a_0 = 1 \]
\[ 2a_2 + a_1 = 1 \]
\[ 3a_3 + a_2 = \frac{1}{2} \]
\[ \ldots \]
\[ (n + 1)a_{n+1} + a_n = \frac{1}{n!} \text{ for } n \geq 0. \]

then \( y \) will satisfy the DE.

The coefficient \( a_0 \) is simply the value of the solution that we will construct, at \( t = 0: a_0 = y(0) \). If we want the solution \( y \) such that \( y(0) = 5 \), for example, we could set \( a_0 = 5 \) and proceed to calculate \( a_1, a_2, \ldots \) as numbers. Instead, we choose to regard \( a_0 \) as an arbitrary constant.

The coefficients \( a_1, a_2, a_3, \ldots \) then all depend upon \( a_0 \). We find

\[ a_1 = 1 - a_0, \]
\[ a_2 = \frac{1-a_1}{2} = \frac{1-(1-a_0)}{2} = \frac{a_0}{2} = \frac{a_0}{2!}, \]
\[ a_3 = \frac{1}{3} \left( \frac{1}{2} - a_2 \right) = \frac{1}{3} \left( \frac{1}{2} - \frac{a_0}{2} \right) = \frac{1-a_0}{3(2)} = \frac{1-a_0}{3!}, \]
\[ a_4 = \frac{1}{4} \left( \frac{1}{3} - a_3 \right) = \frac{1}{4} \left( \frac{1}{3} - \frac{1-a_0}{3(2)} \right) = \frac{a_0}{4!}, \]
\[ a_5 = \frac{1}{5} \left( \frac{1}{4} - a_4 \right) = \frac{1}{5} \left( \frac{1}{4} - \frac{a_0}{4} \right) = \frac{1-a_0}{5(4)} = \frac{1-a_0}{5!}. \]

By examining the pattern that is developing in these formulas for the coefficients, we infer that the general formula for the coefficients \( a_n, n \geq 2 \), is

\[ a_n = \left\{ \begin{array}{ll} \frac{a_0}{(2k)!} & \text{for even indices } n = 2k, k = 1, 2, \ldots \\ \frac{1-a_0}{(2k+1)!} & \text{for odd indices } n = 2k + 1, k = 1, 2, \ldots \end{array} \right. \]

and so

\[ y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_{2k} t^{2k} + a_{2k+1} t^{2k+1} + \ldots \]
\[ = a_0 + (1 - a_0) t + \frac{a_0}{2!} t^2 + \frac{1-a_0}{3!} t^3 + \ldots + \frac{a_0}{(2k)!} t^{2k} + \frac{1-a_0}{(2k+1)!} t^{2k+1} + \ldots \]
1.2.2 Mathematical Justifications I

But is the power series just constructed in fact a solution of the DE? To check, we need to be able to substitute the series in place of \( y \) in the DE. The DE involves both \( y \) and its derivative \( \frac{dy}{dt} \). To be a solution, the series must first of all converge for some interval of values of \( t \), and define a function \( y(t) \) that is differentiable on the interval. Here “on the interval” means, “at all points of the interval”.

It is not difficult to show that the power series for \( y \) given above converges on the open interval \(-\infty < t < \infty\) (this is one of the exercises).

There is a very useful result from the theory of power series:

- If a power series converges on an open interval, then
  - the series defines a differentiable function,
  - the derivative of the function is given by the power series obtained by differentiating, term by term,
  - the series for the derivative converges on the same open interval.

This is all that is needed to justify the operations involved in substituting the series for \( y \) into the DE. The verification that \( y \) is in fact a solution, consists of checking that the series obtained by substituting \( y \) into the left side of the DE, is identical to the series on the right side of the DE.

We leave it as an exercise for the reader to verify that the sum of the series for \( \frac{dy}{dt} \) and for \( y \), is the series \( 1 + t + \frac{1}{2} t^2 + \frac{1}{3!} t^3 + .. + \frac{1}{n!} t^n + ... \)

2 Functions that are analytic at a point

2.1 Definition

A function \( G(x) \) is analytic at \( x = x_0 \) if the Taylor series for \( G \) about \( x_0 \) can be constructed and the Taylor series converges to \( G \) for all \( x \) in some interval \( x_0 - \delta < x < x_0 + \delta \).

Recall that the Taylor series for \( G \) about \( x_0 \) is

\[
\sum_{n=0}^{\infty} \frac{G^{(n)}(x_0)}{n!} (x-x_0)^n = G(x_0) + G'(x_0)(x-x_0) + \frac{G''(x_0)}{2!}(x-x_0)^2 + \frac{G^{(3)}(x_0)}{3!}(x-x_0)^3 + .. + \frac{G^{(n)}(x_0)}{n!}(x-x_0)^n + ...
\]

If \( G \) is analytic at \( x_0 \), then

\[
G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(x_0)}{n!} (x-x_0)^n \text{ for } x_0 - \delta < x < x_0 + \delta.
\]
2.2 Taylor-Maclaurin series are Taylor series about the origin

The Taylor-Maclaurin series for a function \( f(t) \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n = f(0) + f'(0)t + \frac{f''(0)}{2!} t^2 + \ldots + \frac{f^{(n)}(0)}{n!} t^n + \ldots
\]

Saying \( f(t) \) is analytic at \( t=0 \) means that the Taylor-Maclaurin series can be constructed and that the Taylor-Maclaurin series converges to \( f(t) \) for all \( t \) in some interval \( -\delta < t < \delta \), i.e.

\[
f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \text{ for } -\delta < t < \delta.
\]

2.3 Taylor-Maclaurin series for some well-known functions

The following Taylor-Maclaurin series are worked out in most Calculus texts:

\[
\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \ldots + t^n + \ldots, \quad -1 < t < 1
\]

\[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + \ldots, -\infty < t < \infty.
\]

\[
sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \ldots, -\infty < t < \infty.
\]

\[
cos(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots + (-1)^n \frac{t^{2n}}{(2n)!} + \ldots, -\infty < t < \infty.
\]

Since each series converges to the corresponding function for all \( t \) in some interval \( -\delta < t < \delta \), it follows that each of these functions is analytic at \( t = 0 \).

2.4 Results that show a function is analytic at a point

The fundamental way to show that a function is analytic at a point uses Taylor’s theorem with remainder:

\[
G(x) = \sum_{n=0}^{N} G^{(n)}(x_0) \frac{(x-x_0)^n}{n!} + G^{(N+1)}(c) \frac{(x-x_0)^{N+1}}{(N+1)!}.
\]

The last term is called the remainder, in which \( c \) is some point between \( x_0 \) and \( x \). If \( G(x) \) is defined on some interval \( x_0 - \delta < x < x_0 + \delta \) and if it can be shown
that \(\lim_{N \to \infty} G^{(N+1)}(c) \frac{(x-x_0)^{N+1}}{(N+1)!} = 0\) for all \(x\) in this interval, then it follows that \(G\) is analytic at \(x_0\).

There are easier ways. The following results may be used:

*Polynomials, rational functions (ratios of polynomials), all of the trigonometric functions, exponentials and logarithm functions have the property that they are analytic at any point at which they are defined.*

A function given by a power series is analytic at each interior point of the interval of convergence.

If functions \(f(x)\) and \(g(x)\) are both analytic at \(x = x_0\), then so are the functions \(f(x) + g(x)\), \(f(x) - g(x)\), and \(f(x)g(x)\). If in addition \(g(x_0) \neq 0\), then the function \(f(x)/g(x)\) is analytic at \(x = x_0\).

These results mean that *it is often possible to determine that a given expression is analytic at a point, by inspecting the expression.* For example, \(x\) and \(\cos(x)\) are analytic at any point, so the same is true for the product \(x\cos(x)\), and it follows that \(1/(x\cos(x))\) is analytic at any point \(x_0\) such that \(x_0\cos(x_0) \neq 0\).

### 2.5 Some examples of functions that are not analytic

A simple example is \(G_1(x) = 1/x\), which is not analytic at \(x = 0\). However, \(G_1(x)\) is analytic at every \(x_0 \neq 0\): the only point where the function fails to be analytic, is the one point where it is not defined.

The function \(G_2(x) = x^{\frac{5}{2}}\) is not analytic at \(x = 0\) because it does not have a 2nd derivative at this point and so it is not possible to construct the Taylor series about this point. More generally, any function of the form \(G_3(x) = x^r\) where \(r\) is a real constant, is not analytic at \(x = 0\) unless \(r\) is a nonnegative integer. (We follow the convention that \(x^0\) is defined as 1 for all \(x\).)

The function \(G_4(x) = \ln x\) is not analytic at \(x = 0\) or at any point \(x = x_0 < 0\) because it is only defined for \(x > 0\).

The function \(G_5(x) = |x|\) is not analytic at \(x = 0\) because, although it is defined at \(x = 0\), it is not differentiable at \(x = 0\) and so it is not possible to construct the Taylor series about this point.

The function \(G_6(x) = \sum_{n=1}^{\infty} x^n/n^2\) is defined on \(-1 \leq x \leq 1\), the interval of convergence for the power series. Since \(G_6(x)\) is not defined at \(x = 1 + h\) with \(h > 0\), it is not possible to form the limit \(\lim_{h \to 0} (G_6(1 + h) - G_6(1))/h\). Therefore \(G_6(x)\) is not differentiable at \(x = 1\), it is not possible to construct Taylor series about this point, and \(G_6(x)\) is not analytic at \(x = 1\). Similarly, \(G_6\) is not analytic at \(x = -1\), the other endpoint of the interval of convergence.
3 Second order equations with variable coefficients

3.1 Definition

A differential equation of the form,

\[
\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t) y = F(t)
\]

in which \(P(t)\) and \(Q(t)\) are functions of \(t\) and are not both constant, is said to have variable coefficients. The function \(F(t)\) is sometimes called the forcing function, because of the analogy to the mass-spring system.

3.1.1 Division of a DE by the coefficient of the highest derivative

Suppose a DE is given in the form

\[
\tilde{D}(t) \frac{d^2 y}{dt^2} + \tilde{P}(t) \frac{dy}{dt} + \tilde{Q}(t) y = \tilde{F}(t)
\]

where the functions \(\tilde{D}(t), \tilde{P}(t), \tilde{Q}(t), \tilde{F}(t)\), are all defined in some interval and \(\tilde{D}(t)\) is nonzero in this interval. Then dividing the DE by \(\tilde{D}(t)\) gives

\[
\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t) y = F(t),
\]

where \(P(t) = \tilde{P}(t)/\tilde{D}(t)\) \(Q(t) = \tilde{Q}(t)/\tilde{D}(t)\), and \(F(t) = \tilde{F}(t)/\tilde{D}(t)\) for \(t\) in the interval.

3.2 Variable coefficient DEs: a reason to study power series methods

If \(P(t)\) and \(Q(t)\) both happen to be constant (that is, independent of \(t\)) then the DE for \(y(t)\) is said to have constant coefficients.

Power series methods certainly apply to systems with constant coefficients. However, given a constant coefficient DE, it is usually simpler to solve it by one of the more elementary methods available. The main strength of power series methods, is the ability to solve DEs with variable coefficients.

3.2.1 Where DE’s with variable coefficients arise

DEs with variable coefficients arise in physical problems involving Laplace’s equation, the wave equation and the heat equation, especially in problems with geometry other than rectangular.

For example, the electric potential around a direct current high voltage line is described by Laplace’s equation, which is a partial differential equation with the 3 space variables as the independent variables. In this situation, cylindrical
geometry is more appropriate than rectangular and so Laplace’s equation is written in cylindrical coordinates, with the distance to the axis of the wire called the “radial” coordinate $r$. When the method of separation of variables is applied to Laplace’s equation in cylindrical coordinates, that factor of the solution which depends on the radial coordinate $r$ turns out to be the solution of an DE with variable coefficients called Bessel’s equation.

4 Solutions of the homogeneous equation when $t = 0$ is an ordinary point

4.1 Definition: ordinary point

The point $t = 0$ is called an ordinary point for the differential equation

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = 0$$

if both $P(t)$ and $Q(t)$ are analytic at $t = 0$.

4.2 Theorem: General Solution about an Ordinary Point

Suppose that $t = 0$ is an ordinary point for the DE $\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = 0$. Then there is a $\delta > 0$ (or possibly $\delta = \infty$) such that

- the Taylor-Maclaurin series for $P(t)$ converges to $P(t)$ for $-\delta < t < \delta$, and

- the Taylor-Maclaurin series for $Q(t)$ converges to $Q(t)$ for $-\delta < t < \delta$.

The general solution of the DE on the interval $-\delta < t < \delta$, has the form

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 y_1(t) + a_2 y_2(t)$$

where $a_0$ and $a_1$ are arbitrary constants and $y_1(t)$ and $y_2(t)$ are linearly independent. The power series for $y$ converges on the open interval $-\delta < t < \delta$.

Many additional properties of $y$ follow from this last statement. The interval of convergence of the power series for $y$ includes (at least) every point $t$ in $-\delta < t < \delta$, so $y$ is analytic at every point $t$ of the interval $-\delta < t < \delta$, including $t = 0$ (see “Results that show a function is analytic at a point” above). Also, the series for $y$ is absolutely convergent, and may be differentiated term-by-term (see “Mathematical Justifications II”, below).
4.3 The power series method

This is a procedure to evaluate the coefficients $a_n$, $n \geq 2$.

**Step 1.** In place of $y$, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in the left side of the DE, substitute the series

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + \ldots$$

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + (n+1)a_{n+1} t^n + \ldots$$

$$\frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n = 2a_2 + 6a_3 t + \ldots + (n+2)(n+1)a_{n+2} t^n + \ldots$$

**Step 2.** Collect powers of $t$ in the series for the left side of the DE and set the coefficients of each power to zero (write down the equations obtained).

**Step 3.** The equation obtained by setting the coefficient of $t^n$ to zero will relate $a_{n+2}$ to certain of the coefficients $a_{n+1}$, $a_n$, ... This equation is called the **recurrence formula** for the DE.

**Step 4.** Use the recurrence formula to determine $a_2$, $a_3$, $a_4$, ... and (unless it is too difficult) the formula for $a_n$ for general $n$. The coefficients will all be in terms of $a_0$ and $a_1$.

**Step 5.** Substitute the coefficients determined in Step 4, into the form

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + \ldots$$

Rewrite the resulting series in the form

$$y = a_0 y_1(t) + a_1 y_2(t)$$

by taking out $a_0$ and $a_1$ as common factors.

4.3.1 Adopting a new general term

In applying the power series method to a DE with variable coefficients, it will be necessary to adopt a new general term in at least one of the series involved.

Suppose for example the DE has a term $ty$. To find a series to replace $ty$, we multiply the series $y$ given above, by $t$:

$$ty = a_0 t + a_1 t^2 + a_2 t^3 + \ldots + a_n t^{n+1} + \ldots$$

The power $t^{n+1}$ appearing here is inconvenient. It doesn’t match the power $t^n$ in the general term of the series for $\frac{d^2y}{dt^2}$ given above. To fix this problem, we write out the term $a_n t^{n+1}$ and the term that precedes it:

$$ty = a_0 t + a_1 t^2 + a_2 t^3 + \ldots + a_{n-1} t^n + a_n t^{n+1} + \ldots$$
Now we adopt the expression $a_{n-1}t^n$ as the new general term, writing the series for $ty$ as

$$ty = a_0t + a_1t^2 + a_2t^3 + \ldots + a_{n-1}t^n + \ldots$$

The starting value of the index $n$ is now $n = 1$.

In the example above, we effectively replaced $n$ by $n - 1$. Procedures like this are often called change of index, or shift of index. Whatever it is called, the purpose is to make the new general term match the general term of some other series, so that the two series can be easily added together at some later stage in the computation.

Repeating this argument in summation notation, we start with

$$ty = \sum_{n=0}^{\infty} a_n t^{n+1}.$$

Replacing $n$ by $n - 1$ everywhere in the summation gives

$$ty = \sum_{n-1=0}^{\infty} a_{n-1} t^{(n-1)+1} = \sum_{n=1}^{\infty} a_{n-1} t^n.$$

To check that the new general term is correct, write out the first few terms of the series by substituting values of $n$ into the formula for the new general term, Compare these with the first few terms of the series before the change of index.

### 4.3.2 Summation notation versus two dots/general term/three dots notation

An example of two dots/general term/three dots notation is the series

$$y = a_1 + a_1t + a_2t^2 + \ldots + a_nt^n + \ldots$$

and an example of summation notation is the equivalent series

$$y = \sum_{n=0}^{\infty} a_n t^n.$$

An advantage of the two dots/general term/three dots notation is that it can be understood at a glance.

However, summation notation has several advantages. It is compact and unambiguous. The general term and the limits of summation are clearly specified.

A good compromise between the two notations is to “split off” the first few terms of the series, but keep the remaining terms of the series in summation notation; for example,

$$y = a_0 + a_1t + a_2t^2 + \sum_{n=3}^{\infty} a_n t^n.$$

Computer algebra systems may not accept the two dots/general term/three dots notation, but do generally accept summation notation and this compromise notation.
4.3.3 The power series method applied to $\frac{d^2y}{dt^2} - ty = 0$

For this DE, $P(t) = 0$ and $Q(t) = -t$ are both analytic at $t = 0$, so $t = 0$ is an ordinary point. The theorem “General Solution about an Ordinary Point” then shows that on some interval $-\delta < t < \delta$, the general solution of the DE has the form

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n + \ldots$$

In place of $\frac{d^2y}{dt^2}$ in the DE we will substitute the series

$$\frac{d^2y}{dt^2} = 2a_2 + (3)(2)a_3 t + \ldots + (n+2)(n+1)a_{n+2}t^n + \ldots$$

We also need to replace the term $ty$ in the DE with a series. We choose to use the form

$$ty = a_0 t + a_1 t^2 + a_2 t^3 + \ldots + a_{n-1} t^n + \ldots$$

(see “adopting a new general term”, above) because the power of $t$ in the general term of this series matches the power of $t$ in the series used for $\frac{d^2y}{dt^2}$. Substituting these series into the left side of the DE and collecting powers of $t$ gives

$$\frac{d^2y}{dt^2} - ty = 2a_2 + \{(3)(2)a_3 - a_0\} t + \ldots + \{(n+2)(n+1)a_{n+2} - a_{n-1}\} t^n + \ldots$$

Recall that we want $y$ to satisfy $\frac{d^2y}{dt^2} - ty = 0$ for all $t$ in an interval. This will be true if the coefficients $a_n$ satisfy

$$2a_2 = 0, \quad \text{and}$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \quad \text{for} \quad n \geq 1.$$ 

That is, $a_2 = 0$ and

$$(3)(2)a_3 = a_0, \quad (6)(5)a_6 = a_3, \ldots \quad (3k)(3k-1)a_{3k} = a_{3k-3}$$

$$(4)(3)a_4 = a_1, \quad (7)(6)a_7 = a_4, \ldots \quad (3k + 1)(3k)a_{3k+1} = a_{3k-2}$$

$$(5)(4)a_5 = a_2, \quad (8)(7)a_8 = a_5, \ldots \quad (3k + 2)(3k + 1)a_{3k+2} = a_{3k-1}$$

Some of the coefficients are multiples of $a_0$:

$$a_3 = \frac{1}{(3)(2)}a_0,$$

$$a_6 = \frac{1}{(6)(5)}a_3 = \frac{1}{(6)(5)(3)(2)}a_0$$

$$a_{3k} = \frac{1}{(3k)(3k-1)(6)(5)(3)(2)}a_0 \quad \text{for} \quad k = 1, 2, \ldots$$

Some of the coefficients are multiples of $a_1$:

$$a_4 = \frac{1}{(4)(3)}a_1,$$

$$a_7 = \frac{1}{(7)(6)}a_4 = \frac{1}{(7)(6)(4)(3)}a_1,$$

$$a_{3k+1} = \frac{1}{(3k+1)(3k)(7)(6)(4)(3)}a_1 \quad \text{for} \quad k = 1, 2, \ldots$$
Some of the coefficients are multiples of $a_2$. Since $a_2 = 0$,

$$
a_5 = \frac{1}{(5)(4)}a_2 = 0,
$$
$$
a_8 = \frac{1}{(8)(7)}a_5 = 0,
$$
$$
\vdots
$$
$$
a_{3k+2} = \frac{1}{(3k+2)(3k+1)}a_{3k-1} = 0 \text{ for } k = 1, 2, \ldots
$$

Finally, we write down the solution and take out $a_0$ and $a_1$ as common factors:

$$
y = a_0 + a_1 t + a_2 t^2 + \ldots + a_{3k} t^{3k} + a_{3k+1} t^{3k+1} + a_{3k+2} t^{3k+2} + \ldots
$$
$$
= a_0 + \frac{1}{(3)(2)} a_0 t^3 + \ldots + \frac{1}{[(3k)(3k - 1)][(6)(5)][(3)(2)]} a_0 t^{3k} + \ldots
$$
$$
+ a_1 t + \frac{1}{(4)(3)} a_1 t^4 + \ldots + \frac{1}{[(3k+1)(3k)][(7)(6)][(4)(3)]} a_1 t^{3k+1} + \ldots
$$
$$
= a_0 \left[ 1 + \frac{1}{(3)(2)} t^3 + \ldots + \frac{1}{[(3k)(3k - 1)][(6)(5)][(3)(2)]} t^{3k} + \ldots \right]
$$
$$
+ a_1 \left[ t + \frac{1}{(4)(3)} t^4 + \ldots + \frac{1}{[(3k+1)(3k)][(7)(6)][(4)(3)]} t^{3k+1} + \ldots \right].
$$

( Remark: $d^2 y / dt^2 \pm ty = 0$ are known today as Airy’s differential equations. Sir G. B. Airy, in “On the Intensity of Light in the neighborhood of a Caustic”, Trans. Camb. Phil. Soc. VI. (1838), pp.379-402, used an integral of the form $\int_0^\infty \cos \frac{1}{2} \pi (w^3 - mw) dw$, which is easily reduced to the form $\int_0^\infty \cos(t^3 \pm xt) dt$. The latter became known as Airy’s integrals. Stokes later observed that Airy’s integrals satisfy $d^2 v / dx^2 \pm 1/3 xv = 0$, which after the change of variables $x = 3^{1/3} t$, $y(t) = v(3^{1/3} t)$ gives $d^2 y / dt^2 \pm ty = 0$. )

### 4.4 Products of Series and the Recurrence Equation

In the example above, the products $P dy / dt$ and $Qy$ are easy to compute because $P(t)$ vanishes (is identically zero) and $Q(t)$ is simply $-t$. In applying the power series method to more general DEs of the form

$$
\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0
$$

we need a way to compute series for the products $P dy / dt$ and $Qy$.

We will apply the following product-of-series formula:

If $\sum_{j=0}^{\infty} u_j$ and $\sum_{k=0}^{\infty} v_k$ are absolutely convergent series, then

$$
\left( \sum_{j=0}^{\infty} u_j \right) \left( \sum_{k=0}^{\infty} v_k \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} u_{n-k} v_k \right)
$$

where the series on the right side is also absolutely convergent.
To apply this formula to obtain a series for \( P \frac{dy}{dt} \), we first write series for \( P \) and \( \frac{dy}{dt} \) with indices named \( j \) and \( k \), each starting from lower limit 0:

\[
P = \sum_{j=0}^{\infty} p_j t^j, \quad \frac{dy}{dt} = \sum_{k=0}^{\infty} (k+1)a_{k+1} t^k
\]

where \( p_j = P^{(j)}(0)/j! \) for \( j \geq 0 \). Setting \( u_j = p_j t^j \) and \( v_k = (k+1)a_{k+1} t^k \) in the product-of-series formula gives

\[
\left( \sum_{j=0}^{\infty} p_j t^j \right) \left( \sum_{k=0}^{\infty} (k+1)a_{k+1} t^k \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1} t^k \right), \text{ so}
\]

\[
P \frac{dy}{dt} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1} \right) t^n.
\]

Similarly, we write

\[
Q = \sum_{j=0}^{\infty} q_j t^j, \quad y = \sum_{k=0}^{\infty} a_k t^k
\]

where \( q_j = Q^{(j)}(0)/j! \) for \( j \geq 0 \). Then setting \( u_j = q_j t^j \) and \( v_k = a_k t^k \) in the product-of-series formula gives

\[
\left( \sum_{j=0}^{\infty} q_j t^j \right) \left( \sum_{k=0}^{\infty} a_k t^k \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} q_{n-k} a_k t^k \right), \text{ so}
\]

\[
Qy = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} q_{n-k} a_k \right) t^n.
\]

Finally, we add together the series for \( \frac{d^2y}{dt^2} \), \( P \frac{dy}{dt} \) and \( Qy \) to obtain

\[
\frac{d^2y}{dt^2} + P \frac{dy}{dt} + Qy = \sum_{n=0}^{\infty} \left\{ \frac{(n+2)(n+1)a_{n+2}}{a_{n+1}} + \sum_{k=0}^{n} \left( p_{n-k}(k+1)a_{k+1} + q_{n-k}a_k \right) \right\} t^n.
\]

The recurrence equation is therefore

\[
(n + 2)(n + 1)a_{n+2} + \sum_{k=0}^{n} \left( p_{n-k}(k+1)a_{k+1} + q_{n-k}a_k \right) = 0, \quad n = 0, 1, 2, ...
\]

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4.4.1 Mathematical Justifications II

Here we justify the manipulations performed in deriving the recurrence equation.

We use the following properties of power series:

1. If a power series converges on an open interval, then
   - the series is absolutely convergent on this same open interval,
   - the series defines a differentiable function,
   - the derivative of the function is given by the power series obtained by differentiating, term by term,
   - the series for the derivative converges on the same open interval.

2. The theorem “General Solution about an Ordinary Point” tells us that the series
   \[ \sum_{n=0}^{\infty} a_n t^n \]
   converges on the open interval \(-\delta < t < \delta\) and defines a function \(y(t)\), the general solution. Then by properties (2),(3) and (4) above, at each \(t\) in the interval \(-\delta < t < \delta\), \(y(t)\) is differentiable and the derivative is given by the series
   \[ \frac{dy}{dt} = \sum_{n=0}^{\infty} \frac{d}{dt}(a_n t^n) = \sum_{n=1}^{\infty} n a_n t^{n-1} \]
   a power series which itself converges on \(-\delta < t < \delta\).

3. By the same argument but starting this time with the series for \(\frac{dy}{dt}\), it follows that \(\frac{dy}{dt}\) is differentiable, and that
   \[ \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \sum_{n=2}^{\infty} (n)(n-1)a_n t^{n-2} \]
   where this series also converges on the interval \(-\delta < t < \delta\).

4. Since the series for \(\frac{dy}{dt}\), \(\frac{d^2 y}{dt^2}\), \(P\) and \(Q\) are each power series (Taylor-Maclaurin series are a type of power series) and each series converges on the open interval \(-\delta < t < \delta\), by property (1) each of these series is absolutely convergent on this same interval. The applications of the product-of-series formula to obtain series for \(P\frac{dy}{dt}\) and for \(Qy\) are therefore justified, and the resulting series are absolutely convergent.

5. In general, convergent series may be added termwise, so adding the series for \(\frac{d^2 y}{dt^2}\), \(P\frac{dy}{dt}\) and \(Qy\) termwise to form a power series for the left side of the DE, is justified.

6. The power series for the left side of the DE converges for all \(t\) on the open interval \(-\delta < t < \delta\). Since \(y\) is a solution of the DE, the series for the left side of the DE converges to zero. But

   A power series in \(t\) vanishes identically on an interval if and only if the coefficient of each power of \(t^n\), \(n \geq 0\), is zero. It follows that the coefficients \(a_n\), \(n \geq 0\), must obey the recurrence equation.
5 Solutions of the homogeneous equation when
\( t = 0 \) is a not an ordinary point

5.1 Singular points
If the point \( t = 0 \) is not an ordinary point for the DE
\[
\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0
\]
then we say the point \( t = 0 \) is a singular point of the DE.

Recall that \( t = 0 \) is an ordinary point if both of the functions \( P(t) \) and \( Q(t) \) are analytic at \( t = 0 \). Saying that \( t = 0 \) is a singular point is the same as saying that at least one of the functions \( P(t) \), \( Q(t) \) is not analytic at \( t = 0 \).

Some simple examples of functions that are not analytic at \( t = 0 \) are the inverse powers \( \frac{1}{t} \), \( \frac{1}{t^2} \), \( \frac{1}{t^3} \), \( \frac{1}{t^4} \)...

5.2 The Euler equation
The DE of the form
\[
\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta t^2 y = 0
\]
where \( \alpha \) and \( \beta \) are constants, is called the Euler equation. For the Euler equation, if at least one of the constants \( \alpha, \beta \) is not zero, then \( t = 0 \) is a singular point.

The Euler equation can be solved by converting it to an equivalent constant coefficient DE. Letting \( t = e^x \) and \( y(t) = u(x) \), the new unknown function \( u(x) \) satisfies a constant coefficient DE. Therefore there is at least one solution \( u(x) = e^{rx} \) where \( r \) is a constant that may be real or complex. Since \( x = \ln(t) \), the corresponding function \( y(t) \) is \( y(t) = e^{r \ln(t)} \). If \( r \) is real, \( y(t) = e^{r \ln(t)} = t^r \).

These observations motivate looking for solutions of the Euler equation of the form \( y(t) = t^r \). The statement of the following Theorem is modified from Boyce and DiPrima, Elementary Differential Equations and Boundary Value Problems:

**Theorem**
To solve the Euler equation in the interval \( 0 < t < \infty \), substitute \( y = t^r \), and compute the roots \( r_1 \) and \( r_2 \) of the equation
\[
r^2 + (\alpha - 1)r + \beta = 0.
\]
If the roots are real and unequal,
\[
y = c_1 t^{r_1} + c_2 t^{r_2}.
\]
If the roots are real and equal,
\[
y = (c_1 + c_2 \ln t) t^{r_1}.
\]
If the roots are complex, that is, \( r_1 = \mu + i\omega \) and \( r_2 = \mu - i\omega \), then
\[
y = t^\mu \left[ c_1 \cos(\omega \ln t) + c_2 \sin(\omega \ln t) \right].
\]

To solve the Euler equation in the interval \(-\infty < t < 0\), substitute \( y = (-t)^r \). The same three cases arise, and the solution is given by the same formulas except that \((-t)^{r_1} \), \((-t)^{r_2} \), \((-t)^\mu \) and \( \ln (-t) \) appear in place of \( t^{r_1} \), \( t^{r_2} \), \( t^\mu \) and \( \ln t \) respectively.

### 5.3 Definition: regular singular point

Many of the DEs that arise in applications belong to a class of DEs for which \( t = 0 \) is a singular point, and yet the DE has a solution that behaves like a power \( t^r \) for \( t \) near 0, \( t > 0 \). The Euler equation, and Bessel’s equation, are members of this class.

The point \( t = 0 \) is called a **regular singular point** for the DE
\[
\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0,
\]
if it is a singular point, and if both of the following conditions are satisfied:

(i) the function
\[
p(t) = \begin{cases} 
  tP(t) & \text{if } t \neq 0 \\
  \lim_{h \to 0} hP(h) & \text{if } t = 0
\end{cases}
\]
is analytic at \( t = 0 \), and

(ii) the function
\[
q(t) = \begin{cases} 
  t^2Q(t) & \text{if } t \neq 0 \\
  \lim_{h \to 0} h^2Q(h) & \text{if } t = 0
\end{cases}
\]
is analytic at \( t = 0 \).

If \( t = 0 \) is a regular singular point for the DE, then since \( p \) and \( q \) are each analytic at \( t = 0 \), it follows that for small but nonzero \( t \),
\[
P(t) = \frac{p(t)}{t} = \frac{p(0)}{t} + p'(0) + \frac{p''(0)}{2}t + \cdots + \frac{p^{(n)}(0)}{n!}t^{n-1} + \cdots
\]
and
\[
Q(t) = \frac{q(t)}{t^2} = \frac{q(0)}{t^2} + \frac{q'(0)}{t} + \frac{q''(0)}{2} + \frac{q^{(3)}(0)}{3!}t + \cdots + \frac{q^{(n)}(0)}{n!}t^{n-2} + \cdots
\]
The only singular (i.e. nonanalytic) behaviors that \( P \) and \( Q \) may exhibit, are:
- If \( p(0) \neq 0 \), then \( P(t) \) “blows up” like \( p(0)/t \) for \( t \) near 0.
- If \( q(0) \neq 0 \), then for \( t \) near 0, then \( Q(t) \) “blows up” like \( q(0)/t^2 \).
- If \( q(0) = 0 \) and \( q'(0) \neq 0 \), then \( Q(t) = q(t)/t^2 \) “blows up” like \( q'(0)/t \).

When a DE has \( t = 0 \) as a regular singular point, we often write the DE as
\[
\frac{d^2y}{dt^2} + \frac{p(t)}{t} \frac{dy}{dt} + \frac{q(t)}{t^2} y = 0.
\]
The general theory for DEs of this type provides different solutions \( y \) in the separate cases \( t > 0 \) and \( t < 0 \). The case \( t = 0 \) in the DE does not arise because there is no attempt to provide a single solution that is valid on an interval including the origin. It may happen that formulas derived for the solutions of a specific DE, turn out to be meaningful on an interval including the origin but this is not to be expected in general.

5.4 An example with a regular singular point

Consider the DE

\[
d^2y \over dt^2 + {2 \over t(t^2 + 1)} \overset{dy}{dt} + {\sin(t) \over t^3} y = 0.
\]

Since \( P(t) = {2 \over t(t^2 + 1)} \) is not analytic at \( t = 0 \), the point \( t = 0 \) is not an ordinary point for the DE and so is a singular point.

To see if \( t = 0 \) is a regular singular point, we first check condition (i). We calculate

\[
\lim_{h \to 0} hP(h) = \lim_{h \to 0} \left( {2 \over h(1 + h^2)} \right) = 2,
\]

and consider

\[
p(t) = \begin{cases} 
  t \left( {2 \over t^2 + 1} \right) & \text{if } t \neq 0 \\
  2 & \text{if } t = 0
\end{cases} = {2 \over 1 + t^2}, \text{ for all } t.
\]

Since \( p(t) \) is given by a rational function (the ratio of two polynomials) with a denominator that is nonzero at \( t = 0 \), it follows that \( p(t) \) is analytic at \( t = 0 \); see “Results that show a function is analytic at a point”, above.

To check condition (ii), we calculate

\[
\lim_{h \to 0} h^2Q(h) = \lim_{h \to 0} h^2 \left( {\sin(h) \over h^3} \right) = \lim_{h \to 0} {\sin h \over h} = 1,
\]

and consider

\[
q(t) = \begin{cases} 
  t^2 \sin(t) \over t^2 & \text{if } t \neq 0 \\
  1 & \text{if } t = 0
\end{cases} = \begin{cases} 
  \sin(t) \over t & \text{if } t \neq 0 \\
  1 & \text{if } t = 0
\end{cases} = \sum_{n=0}^{\infty} (-1)^n {t^{2n} \over (2n + 1)!},
\]

for \(-\infty < t < \infty \). Since \( q(t) \) is given by a power series and \( t = 0 \) is an interior point of the interval of convergence (which is the , \( q(t) \) is analytic at \( t = 0 \); see “Results that show a function is analytic at a point”, above.

It follows that \( t = 0 \) is a regular singular point for the DE.

The basic result in the general theory developed by Frobenius is the following Theorem.
5.5 Theorem: General solution in intervals right and left of a regular singular point

Suppose \( t = 0 \) is a regular singular point for the DE

\[
\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0.
\]

Then we may rewrite the DE in the form

\[
\frac{d^2y}{dt^2} + \frac{p(t)}{t} \frac{dy}{dt} + \frac{q(t)}{t^2} y = 0
\]

where \( p(t) \) and \( q(t) \) are each analytic at \( t = 0 \). There is a \( \delta > 0 \) (possibly \( \delta = \infty \)) such that

- the Taylor-Maclaurin series for \( p(t) \) converges to \( p(t) \) for \(-\delta < t < \delta\), and
- the Taylor-Maclaurin series for \( q(t) \) converges to \( q(t) \) for \(-\delta < t < \delta\).

In the interval \( 0 < t < \delta \), the general solution of the DE may be expressed as

\[
y = c_1 y_1 + c_2 y_2
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants and \( y_1 \) and \( y_2 \) are linearly independent solutions. If the roots \( r_1, r_2 \) of the indicial equation \( r^2 + (p(0) - 1)r + q(0) = 0 \) are real, and \( r_1 \geq r_2 \), then \( y_1 \) and \( y_2 \) are of the forms

\[
y_1(t) = t^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n t^n \right],
\]

\[
y_2(t) = \begin{cases} 
  t^{r_2} \left[ 1 + \sum_{n=1}^{\infty} b_n t^n \right] & \text{if } r_1 - r_2 \text{ not an integer or zero}, \\
  y_1(t) \ln t + t^{r_2} \sum_{n=1}^{\infty} b_n t^n & \text{if } r_1 = r_2, \\
  a y_1(t) \ln t + t^{r_2} \left[ 1 + \sum_{n=1}^{\infty} b_n t^n \right] & \text{if } r_1 - r_2 = N, \text{ a positive integer}.
\end{cases}
\]

If the roots of the indicial equation are complex, that is, \( r_1 = \mu + i\omega \) and \( r_2 = \mu - i\omega \) where \( \mu \) and \( \omega \) are real, then

\[
y_1(t) = t^{\mu} \cos(\omega \ln t) \left[ 1 + \sum_{n=1}^{\infty} a_n t^n \right],
\]

\[
y_2(t) = t^{\mu} \sin(\omega \ln t) \left[ 1 + \sum_{n=1}^{\infty} b_n t^n \right].
\]

In the interval \(-\delta < t < 0\), the general solution \( y \) of the DE may be expressed in terms of two linearly independent solutions \( y_1 \) and \( y_2 \) as above, except that in the forms for \( y_1 \) and \( y_2 \), \((-t)^{r_1}, (-t)^{r_2}, (-t)^\mu \) and \( \ln (-t) \) appear in place of \( t^{r_1}, t^{r_2}, t^\mu \) and \( \ln t \) respectively.

The coefficients \( a_n, b_n \), and the constant \( a \) can be determined by substituting the form of the solutions \( y_1 \) and \( y_2 \) in the DE. The constant \( a \) may turn out to be zero. Each of the series \( \sum_{n=1}^{\infty} a_n t^n, \sum_{n=1}^{\infty} b_n t^n \), converges for \(-\delta < t < \delta\).
5.6 The Method of Frobenius

This is a method to find at least one solution of a DE with a regular singular point, in the the case the indicial equation has real roots. The method also determines the indicial equation itself.

In the Method of Frobenius, it is traditional to multiply the DE by the factor $t^2$ before substituting in series. Multiplying the DE by $t^2$ makes no difference at all in the solution $y$. It is done simply to allow writing $1$, $t$, $t^2$ and $t^r$ as factors during the intermediate computation instead of the less convenient $1/t^2$, $1/t$, $1$ and $t^{r-2}$.

**Step 1.** Write the DE in the form,

$$t^2 \frac{d^2 y}{dt^2} + p(t) \left( \frac{dy}{dt} \right) + q(t)y = 0.$$ 

In place of $y$, $t \frac{dy}{dt}$ and $t^2 \frac{d^2 y}{dt^2}$ in the left side of the DE, substitute the series

$$y = t^r \left[ 1 + a_1 t + a_2 t^2 + a_3 t^3 + .. + a_n t^n + ... \right]$$

$$t \frac{dy}{dt} = t^r \left[ r + (1 + r)a_1 t + (2 + r)a_2 t^2 + (3 + r)a_3 t^3 + .. + (n + r)a_n t^n + ... \right]$$

$$t^2 \frac{d^2 y}{dt^2} = t^r \left[ r(r-1) + (1 + r)(r)a_1 t + (2 + r)(1 + r)a_2 t^2 + (3 + r)(2 + r)a_3 t^3 + .. + (n + r)(n + r - 1)a_n t^n + ... \right]$$

**Step 2.** Collect powers of $t$ and set the coefficients of each power to zero (write down the equations obtained).

**Step 3.** The equation obtained by setting the coefficient of the lowest power of $t$ to zero, is the indicial equation. Solve the indicial equation to find the roots $r_1$ and $r_2$, where $r_1$ is the larger root.

**Step 4.** The equations obtained by setting the coefficients of the higher powers of $t$ to zero, are called the recurrence equations. These will relate $a_n$ to certain of the coefficients $a_{n-1}$, ..., $a_1$.

Starting with $n = 1$, use the recurrence equations with $r$ replaced by $r_1$ to determine $a_1$, $a_2$, $a_3$, $a_4$, ... and (unless it is too difficult) the formula for $a_n$ for general $n$.

**Step 5.** Substitute the coefficients determined in Step 4, into the form

$$y = y_1 = t^{r_1} \left[ 1 + a_1 t + a_2 t^2 + a_3 t^3 + .. + a_n t^n + ... \right]$$

5.7 The Method of Frobenius applied to $\frac{d^2 y}{dt^2} + \frac{1}{t} y = 0$

For this DE, it is readily shown that $t = 0$ is a regular singular point, and the functions $p$ and $q$ are $p = 0$ and $q = t$. 

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To apply the Method of Frobenius, we write the DE in the “traditional” form

\[ t^2 \frac{d^2 y}{dt^2} + ty = 0. \]

From the series given above,

\[ t^2 \frac{d^2 y}{dt^2} = t^r \left[ (r)(r - 1) + (1 + r)(r)a_1 t + \ldots + (n + r)(n + r - 1)a_n t^n + \ldots \right] \]

\[ ty = t^r \left[ t + a_1 t^2 + \ldots + a_n t^{n+1} + \ldots \right]. \]

The general term of this series for \( ty \) doesn’t match the general term in the series for \( t^2 \frac{d^2 y}{dt^2} \), so (see “Adopting a new general term”, above) we change index to obtain

\[ ty = t^r \left[ t + a_1 t^2 + \ldots + a_{n-1} t^n + \ldots \right]. \]

Substituting the series for \( ty \) and for \( t^2 \frac{d^2 y}{dt^2} \) in the left side of the DE gives

\[ t^2 \frac{d^2 y}{dt^2} + ty = t^r \left[ (r)(r - 1) + (1 + r)(r)a_1 t + (2 + r)(1 + r)a_2 t^2 + \ldots \right] + \]

\[ + (n + r)(n + r - 1)a_n t^n + \ldots \]

\[ + t^r \left[ t + a_1 t^2 + \ldots + a_{n-1} t^n + a_n t^{n+1} + \ldots \right] \]

\[ = t^r \left[ \left( r(r - 1) + \{(1 + r)(r)a_1 + 1\} \right) t \right. \]

\[ + \{(2 + r)(1 + r)a_2 + a_1\} t^2 \]

\[ + \{(n + r)(n + r - 1)a_n + a_{n-1}\} t^n + \ldots \]

Equating the coefficients of the different powers of \( t \) to zero gives:

\[ r(r - 1) = 0 \quad \text{(the indicial equation)} \]

\[ (1 + r)(r)a_1 + 1 = 0 \quad \text{(1st recurrence equation)} \]

\[ (n + r)(n + r - 1)a_n + a_{n-1} = 0 \quad \text{for } n = 2, 3, \ldots \]

Solving the indicial equation gives the roots \( r_1 = 1 \) and \( r_2 = 0 \) (\( r_1 \) is the larger root).

Setting \( r = r_1 = 1 \) in the recurrence equations gives

\[ (2)(1)a_1 + 1 = 0 \]

\[ (n + 1)(n)a_n + a_{n-1} = 0 \quad \text{for } n = 2, 3, \ldots \]

Solving the recurrence equations gives

\[ a_1 = -\frac{1}{2} \]

\[ a_2 = -\frac{1}{(3)(2)} a_1 = -\frac{1}{(3)(2)} \left( -\frac{1}{2} \right) = \frac{1}{4(3)} \]

\[ a_3 = -\frac{1}{(4)(3)} a_2 = -\frac{1}{(4)(3)(2)} = -\frac{1}{4(3)(2)} \]

\[ = -\frac{1}{4(3)^2} \]
\[ a_4 = -\frac{1}{(5)(4)}a_3 = + \frac{1}{(5)(4)(3)(2)(2)} = \frac{1}{5(4)!} \]
\[ a_n = -\frac{1}{(n+1)(n)}a_{n-1} = (1)^2 \frac{1}{(\alpha+1)(\alpha)(\alpha-1)}a_{n-2} = \ldots = (1)^n \frac{1}{(\alpha+1)(\alpha)!}. \]

Substituting the coefficients obtained into the form for \( y_1 \) gives
\[ y_1 = t^{r_1} \left[ 1 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + \ldots \right] = t^1 \left[ 1 - \frac{1}{2} t + \frac{1}{(2)(4)} t^2 - \frac{1}{(4)(6)} t^3 + \ldots + (-1)^n \frac{1}{(\alpha+1)(\alpha)!} t^n + \ldots \right] \]

### 5.8 The second solution \( y_2 \) for \( \frac{d^2 y}{dt^2} + \frac{1}{t} y = 0 \)

The difference between the roots of the indicial equation is \( r_1 - r_2 = 1 \), a positive integer. From the theorem on the form of the general solution, \( y_2 \) is of the form
\[ y_2 = ay_1(t) \ln t + t^{r_2} \left[ 1 + b_1 t + b_2 t^2 + \ldots + b_n t^n + \ldots \right] \]
where the constant \( a \) and the coefficients \( b_n \) for \( n \geq 1 \) are to be determined by substitution into the DE.

To give the series containing the unknown coefficients a name, we define
\[ w = 1 + b_1 t + b_2 t^2 + \ldots + b_n t^n + \ldots. \]
Then since \( t^{r_2} = t^0 = 1 \), we have
\[ y_2 = ay_1(t) \ln t + w. \]

We calculate using the product rule:
\[ \frac{dy_2}{dt} = a \frac{dy_1}{dt} \ln t + ay_1 \frac{1}{t} + \frac{dw}{dt}, \]
\[ \frac{d^2 y_2}{dt^2} = a \frac{d^2 y_1}{dt^2} \ln t + 2a \frac{dy_1}{dt} \frac{1}{t} + ay_1 \left( \frac{-1}{t^2} \right) + \frac{d^2 w}{dt^2}. \]

Then setting \( y_2 \) into the DE in the form \( t^2 \frac{d^2 y}{dt^2} + ty = 0 \) gives
\[ at^2 \frac{d^2 y_1}{dt^2} \ln t + 2ta \frac{dy_1}{dt} - ay_1 + t^2 \frac{d^2 w}{dt^2} + aty_1 \ln t + tw = 0 \]
\[ a \left[ t^2 \frac{d^2 y_1}{dt^2} + ty_1 \right] \ln t + 2ta \frac{dy_1}{dt} - ay_1 + t^2 \frac{d^2 w}{dt^2} + tw = 0 \]
But since \( y_1 \) obeys \( t^2 \frac{d^2 y_1}{dt^2} + ty_1 = 0 \), the DE with \( y = y_2 \) simplifies to
\[
t^2 \frac{d^2 w}{dt^2} + tw + 2ta \frac{dy_1}{dt} - ay_1 = 0.
\]

At this point we start to work with series termwise. The coefficients \( b_1, b_2, \ldots \) and the constant \( a \) are unknown, but the coefficients \( a_1, a_2, \ldots \) are known since the solution \( y_1 \) was found previously. We compute
\[
tw = t + b_1 t^2 + \ldots + b_{n-1} t^n + \ldots
\]
\[
\frac{d^2 w}{dt^2} = 2b_2 + (3)(2)b_3 t + \ldots + n(n-1)b_n t^{n-2} + \ldots
\]
\[
t^2 \frac{d^2 w}{dt^2} = 2b_2 t^2 + (3)(2)b_3 t^3 + \ldots + n(n-1)b_n t^n + \ldots
\]
\[
y_1 = t + a_1 t^2 + a_2 t^3 + \ldots + a_{n-1} t^n + \ldots
\]
\[
\frac{t}{t} \frac{dy_1}{dt} = t + 2a_1 t^2 + 3a_2 t^3 + \ldots + na_{n-1} t^n + \ldots
\]
\[
t^2 \frac{d^2 w}{dt^2} + tw + 2a(t \frac{dy_1}{dt}) - ay_1 = \{1 + a\} t
\]
\[
+ \{2b_2 + b_1 + 3aa_1\} t^2 + \ldots
\]
\[
+ \{(n)(n-1)b_n + b_{n-1} + a(2n-1)a_{n-1}\} t^n + \ldots
\]

By equating the coefficients of the powers of \( t \) to zero, immediately
\[
a = -1
\]
and then (using this value in the remaining equations)
\[
(n)(n-1)b_n + b_{n-1} - (2n-1)a_{n-1} = 0 \text{ for } n \geq 2.
\]

For \( n = 2 \),
\[
2b_2 + b_1 - 3a_1 = 0.
\]

Now we observe that \( b_1 \) may be chosen arbitrarily. For simplicity, we choose \( b_1 = 0 \). (Remark: If \( b_1 \) is left as an arbitrary constant, it will turn out that the solution \( y_2 \) includes a term \( b_1 y_1 \). Since the general solution already includes a constant multiple of \( y_1 \), the multiple \( b_1 y_1 \) is not needed and we may as well take \( b_1 = 0 \).)

We calculate
\[
b_2 = \frac{3}{2} a_1 = -\frac{3}{4},
\]
\[
b_3 = \frac{1}{(3)(2)} \left[ -b_2 + 5a_2 \right] = \frac{1}{6} \left[ \frac{3}{4} + \frac{5}{2} \right] = \frac{7}{36},
\]
and so
\[
y_2 = -y_1 (t) \ln t + \left[ 1 - \frac{3}{4} t^2 + \frac{7}{36} t^3 + \ldots \right].
\]

To determine the general formula for the coefficients \( b_n \) takes some additional
effort. The idea is to use the general formula for the coefficients $a_n$ as the basis for determining the formula for the coefficients $b_n$.

We observe that the recurrence equation for the coefficients $b_n$ resembles the equation

$$(n + 1)(n)a_n + a_{n-1} = 0 \text{ for } n = 2, 3, ...$$

To make the equations look even more similar, we shift index:

$$(n)(n-1)a_n - 1 + a_{n-2} = 0 \text{ for } n = 3, 4, ...$$

We expect, therefore, that the general formula for the coefficients $b_n$ will resemble the general formula for the coefficients $a_{n-1}$, which is known. If the formulas resemble each other, then the formula for the ratio $\frac{b_n}{a_{n-1}}$, should be simpler. This idea leads us to **define**

$$\beta_n = \frac{b_n}{a_{n-1}} \text{ for each } n \geq 2$$

and to look for a general formula for $\beta_n$.

To obtain a recurrence equation for the $\beta_n$’s, we replace $b_n$ with $\beta_n a_{n-1}$ and $b_{n+1}$ with $\beta_{n+1} a_{n-2}$ in the recurrence equation for the $b_n$’s: this gives

$$(n)(n-1)\beta_n a_{n-2} = -\beta_{n-1} a_{n-2} + (2n - 1)a_n - 1,$$

But $a_n = -\frac{(n-1)(n-2)}{a_{n-1}}$, so

$$(n)(n-1)\beta_n a_{n-1} = \beta_{n-1} a_{n-2} + (2n - 1)a_n - 1.$$

Dividing by $(n)(n-1)a_{n-1}$ and simplifying gives

$$\beta_n = \beta_{n-1} + \frac{2n - 1}{n(n - 1)}, \text{ for } n \geq 3.$$

We start with $\beta_2 = b_2/a_1 = -\frac{3}{4}/(-\frac{1}{2}) = \frac{3}{2}$ and then calculate

$$\beta_3 = \frac{3}{2} + \frac{5}{6},$$

$$\beta_4 = \frac{3}{2} + \frac{5}{6} + \frac{7}{12},$$

$$\beta_n = \frac{3}{2} + \frac{5}{6} + \ldots + \frac{2n - 1}{n(n-1)}$$

$$= \sum_{j=2}^{n} \frac{2j - 1}{j(j-1)}, \text{ for } n \geq 2.$$

so then the general formula for $b_n$ is

$$b_n = a_{n-1}\beta_n = (-1)^{n-1} \frac{1}{n((n-1)!)^2} \sum_{j=2}^{n} \frac{2j - 1}{j(j-1)} \text{ for } n \geq 2.$$

It follows that the solution $y_2$ is given by

$$y_2(t) = -y_1(t) \ln t + w, \text{ where }$$

$$w = 1 - \frac{3}{4} t^2 + \frac{7}{36} t^3 + \ldots + (-1)^{n-1} \frac{1}{n((n-1)!)^2} \sum_{j=2}^{n} \frac{2j - 1}{j(j-1)} t^n + \ldots$$

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6 Exercises for Chapter X

1a) Differentiate the series

\[ y = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \]

term-by-term.

1b) Compare both the series for \( y \) and the series for \( \frac{dy}{dt} \) in part a) to the Taylor-Maclaurin series for well-known functions in section 2.3. What well-known formula is established in part a)?

2a) Substitute

\[ y = a_0 + a_1 t + ... + a_{n-1} t^{n-1} + a_n t^n + a_{n+1} t^{n+1} + ... \]

into the DE \( \frac{dy}{dt} - ty = 0 \) and derive an equation (the recurrence equation) for the coefficients.

2b) Find the coefficients \( a_0, a_1, \) and \( a_2 \) for the power series solution of the initial value problem \( \frac{dy}{dt} - ty = 0, \ y(0) = 5 \).

3a) Verify that \( a_k = \frac{1}{k!}, k \geq 0, \) satisfies the recurrence equation

\[(n+1)a_{n+1} - a_n = 0 \] for \( n \geq 1 \).

3b) Let \( y(t) \) be defined by the power series \( y = \sum a_n t^n \) where the coefficients \( a_n \) are as in part a). Differentiate the series for \( y(t) \) term-by-term.

3c) Change index in the series for \( \frac{dy}{dt} \), so that in the new series, the power of \( t \) in the general term is the same as the new summation index.

3d) By inspecting the series for \( \frac{dy}{dt} \) and for \( y \), find a DE for which \( y \) is a solution.

4. Verify that

\[ a_n = \begin{cases} \frac{a_0}{(2k)!} & \text{for even indices } n = 2k, k = 1, 2, ... \\ \frac{1-a_0}{(2k+1)!} & \text{for odd indices } n = 2k + 1, k = 1, 2, ... \end{cases} \]

satisfies the recurrence equation

\[(n+1)a_{n+1} + a_n = \frac{1}{n!} \] for \( n \geq 0 \).

5. a) Show that for any constant \( a_0 \), the series

\[ a_0 + (1-a_0)t + \frac{a_0}{2!} t^2 + \frac{1-a_0}{3!} t^3 + ... + \frac{a_0}{(2k)!} t^{2k} + \frac{1-a_0}{(2k+1)!} t^{2k+1} + ... \]
is absolutely convergent for all $t$. (Hint: Compare the absolute value of the $t^n$ term of this series, to $\frac{M}{n!}|t|^n$, where $M = \max\{|a_0|, |1 - a_0|\}$.)

b) Show that the power series in part a) satisfies the differential equation

$$\frac{dy}{dt} + y = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^n}{n!} + \ldots$$

c) Write the solution from part a) in the form, $y = a_0y_1(t) + y_0(t)$, where $y_1(t)$ and $y_0(t)$ are each defined by series.

6. Use the power series method to find the general solution about the origin, of

$$\frac{d^2y}{dt^2} + y = 0.$$ 

Identify the functions $y_1(t)$ and $y_2(t)$ by comparing their series to the Taylor series for some well known functions in section 2.3.

7. Show that the theorem on the general solution about an ordinary point applies to the DE

$$\frac{d^2y}{dt^2} + \frac{1}{1-t} \frac{dy}{dt} + \sin(t)y = 0.$$ 

Determine $\delta$ such that the general solution $y$ on the interval $-\delta < t < \delta$ may be obtained in the form of a power series about $t = 0$.

8a) Find the general solution of

$$\frac{d^2y}{dt^2} + ty = 0$$

in terms of a power series about the origin.

b) On what interval is the solution valid?

9. Find the general solution of

$$\frac{d^2y}{dt^2} - 2t \frac{dy}{dt} - 2y = 0$$

in terms of a power series about the origin.

10. Find the general solution in terms of power series about $x = 2$, of

$$\frac{d^2u}{dx^2} + x^2 \frac{du}{dx} + (2 - x)u = 0$$

11. Solve:

$$(t^2 + 1) \frac{d^2y}{dt^2} + y = 3t, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0.$$
(Remark: Here the power series method should be applied to the DE in the form given, \textit{without} dividing by \((t^2 + 1)\). Also, this is not a homogeneous equation: the coefficient of \(t\) on the left, must be set equal to 3, the coefficient of the \(t\) on the right.)

12a) Show that the function

\[
f(t) = \begin{cases} \sin t & \text{if } t \neq 0 \\ \frac{\sin t}{t} & \text{if } t = 0 \end{cases}
\]

may also be expressed

\[
f(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k+1)!}, \text{ for } -\infty < t < \infty.
\]

12b) Differentiate the series for \(f\) term-by-term to obtain the general formula for the \(n\)-th derivative \(f^{(n)}(0), n \geq 0\).  
12c) Show the series for \(f\) and the Taylor-Maclaurin series for \(f\) are in fact the same series.  
12d) Show that \(f\) is analytic at \(t = 0\), by using the fact that \(f(t)\) is given by its Taylor-Maclaurin series.

13. Determine whether \(t = 0\) is a regular singular point for the differential equation

\[
\frac{d^2 y}{dt^2} - t \frac{dy}{dt} + t^2 y = 0
\]

14. Determine whether \(t = 0\) is a regular singular point for the differential equation

\[
t^2 \frac{d^2 y}{dt^2} + t(t+1) \frac{dy}{dt} + 2y = 0
\]

15. Use the method of Frobenius to find one solution of

\[
t^2 \frac{d^2 y}{dt^2} - t \frac{dy}{dt} + y = 0.
\]

16. Use the method of Frobenius to find one solution of

\[
t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + t^2 y = 0.
\]

( Remark: \(t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - \nu^2)y = 0\), is called \textbf{Bessel’s equation of order} \(\nu\) so this exercise finds a solution of Bessel’s equation of order \(\nu = 0\). If the solution \(y_1\) obtained by the method of Frobenius is adjusted so that \(y_1(0) = 1\), then in fact \(y_1(t) = J_0(t)\), the Bessel function of the first kind of order zero.

The authoritative source of information about Bessel functions, is G. N. Watson, \textit{A Treatise on the Theory of Bessel Functions}, Cambridge Univ. Press (1922; first paperback edition 1966) )
7 Solutions to Odd Numbered Exercises

1a. 
\[ \frac{dy}{dt} = \sum_{n=0}^{\infty} (-1)^n (2n + 1) \frac{t^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n (2n + 1) \frac{t^{2n}}{(2n+1)[(2n)(2n-1)...(1)]} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \]

1b. \( \frac{d}{dt} \sin(t) = \cos(t) \).

3a. Setting \( k = n + 1 \) in the formula \( a_k = \frac{1}{k!} \), gives \( a_{n+1} = \frac{1}{(n+1)!} \). Similarly, setting \( k = n \) in the formula \( a_k = \frac{1}{k!} \), gives \( a_n = \frac{1}{n!} \). Then

\[ (n + 1)a_{n+1} - an = (n + 1) \frac{1}{(n+1)!} - \frac{1}{n!} = \frac{1}{n!} - \frac{1}{n!} = 0. \]

3b. \( \frac{dy}{dt} = \sum_{n=1}^{\infty} n \frac{a_{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} \)

3c. Let \( n = k \), where \( k \) will be the new summation index. Then \( n = k+1 \), and

\[ \frac{dy}{dt} = \sum_{k+1}^{\infty} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}. \]

3d. The series for \( \frac{dy}{dt} \) is the same as the series for \( y \): calling the index \( k \) rather than \( n \) doesn’t make a difference. Therefore \( \frac{dy}{dt} = y \). This is the DE.

5a. Following the hint,

\[ |a_0| + |(1 - a_0)t| + \frac{|a_0|}{2} |t^2| + \frac{|1 - a_0| |t^3|}{3!} + \frac{|a_0|}{(2k)!} |t^{2k}| + \frac{1 - a_0}{(2k + 1)!} |t^{2k+1}| + \ldots \]

\[ = |a_0| + |(1 - a_0)||t| + \frac{|a_0|}{2} |t^2| + \frac{|1 - a_0| |t^3|}{3!} + \frac{|a_0|}{(2k)!} |t^{2k}| + \frac{1 - a_0}{(2k + 1)!} |t^{2k+1}| + \ldots \leq M + M|t| + \frac{M}{2} |t^2| + \frac{M}{3!} |t^3| + \ldots + \frac{M}{(2k)!} |t^{2k}| + \frac{M}{(2k + 1)!} |t^{2k+1}| + \ldots \]

where \( M = \max(|a_0|, |1 - a_0|) \). This last series is a positive term series, which converges for all \( t \). This can be shown by the ratio test.

It follows by the comparison test that the series for \( y \) is absolutely convergent for all \( t \).

5b. we may differentiate termwise and the resulting series

\[ y = a_0 + (1 - a_0)t + \frac{a_0}{2} t^2 + \frac{1 - a_0}{3!} t^3 + \ldots + \frac{a_0}{(2k)!} t^{2k} + \frac{1 - a_0}{(2k + 1)!} t^{2k+1} + \frac{1 - a_0}{(2k + 2)!} t^{2k+2} + \ldots \]

\[ \frac{dy}{dt} = (1 - a_0) + a_0 t + \frac{a_0}{2!} t^2 + \ldots + \frac{a_0}{(2k - 1)!} t^{2k-1} + \frac{1 - a_0}{(2k)!} t^{2k} + \frac{a_0}{(2k+1)!} t^{2k+1} + \ldots \]
\[
\frac{dy}{dt} + y = (1 - a_0) + a_0 + a_0 t + (1 - a_0) t + \frac{1}{2!} a_t^2 + \frac{a_0}{2} t^2 + \ldots \\
\quad + \frac{1 - a_0}{(2k)!} t^{2k} + \frac{a_0}{(2k)!} t^{2k} + \ldots \\
= 1 + t + \frac{1}{2!} t^2 + \ldots + \frac{1}{(2k)!} t^{2k} + \ldots
\]

5c. By grouping together all terms with \(a_0\) as a factor,

\[
y = a_0 \left[ 1 - t + \frac{1}{2} t^2 - \frac{1}{3!} t^3 + \ldots + \frac{1}{(2k)!} t^{2k} - \frac{1}{(2k+1)!} t^{2k+1} + \ldots \right] \\
\quad + \left[ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots + \frac{t^{2k+1}}{(2k+1)!} + \ldots \right]
\]

7. This DE is of the form \(\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0\) where \(P(t) = \frac{1}{1-t}\) and \(Q(t) = \sin(t)\). These are “well-known” functions, both analytic at \(t = 0\). The Taylor-Maclaurin series for \(P(t) = \frac{1}{1-t}\) converges to \(\frac{1}{1-t}\) for \(-1 < t < 1\). The Taylor-Maclaurin series for \(Q(t) = \sin t\) converges to \(\sin t\) for \(-\infty < t < \infty\).

On the interval \(-\delta < t < \delta\), where \(\delta = 1\) both of these series converge to their respective functions.

Then by the basic theorem, the general solution of the DE has the form of a power series \(\sum_{n=0}^{\infty} a_n t^n\) which converges for \(-\delta < t < \delta\).

9. We compute:

\[
y = \sum_{n=0}^{\infty} a_n t^n, \quad \frac{dy}{dt} = \sum_{n=1}^{\infty} na_n t^{n-1}, \\
\frac{d^2y}{dt^2} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} t^m \\
\quad = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n \\
-2t \frac{dy}{dt} = -2t \sum_{n=1}^{\infty} na_n t^{n-1} = \sum_{n=1}^{\infty} \left((-2n)a_n t^n \right)
\]

Then

\[
\frac{d^2y}{dt^2} - 2t \frac{dy}{dt} - 2y = 2a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2na_n - 2a_n\right] t^n
\]

which we want to equal zero. If the coefficients satisfy the recurrence equations

\[
2a_2 - 2a_0 = 0
\]

\[
(n+2)(n+1)a_{n+2} - 2(n+1)a_n = 0, \quad \text{for } n \geq 1
\]

then \(y\) will be a solution. Therefore

\[
a_2 = a_0, \quad a_4 = \frac{2}{5} a_2 = \frac{1}{2} a_0, \quad a_6 = \frac{2}{5} a_4 = \frac{1}{10} a_0, \ldots \\
a_{2k} = \frac{1}{(2k)!} a_0 \text{ for } k \geq 1
\]

\[
a_3 = \frac{2}{3} a_1, \quad a_5 = \frac{2}{5} a_3 = \frac{2^2}{(5)(3)} a_1, \quad a_7 = \frac{2}{5} a_5 = \frac{2^3}{(7)(5)(3)} a_1, \ldots \\
a_{2k+1} = \frac{2^k}{(2k+1)(2k-1) \ldots (3)} a_1 \text{ for } k \geq 1
\]
and the general solution \( y \) is given by
\[
y = a_0 \left[ 1 + t^2 + \frac{1}{2} t^4 + \ldots + \frac{1}{2^k} t^{2k} + \ldots \right] \\
+ a_1 \left[ t + \frac{2}{3} t^3 + \frac{2^2}{(5)(3)} t^5 + \ldots + \frac{2^k}{(2k+1)2k-1 \cdot \ldots \cdot 3} t^{2k+1} + \ldots \right]
\]

11. In the usual manner,
\[
\frac{d^2 y}{dt^2} = \sum_{n=2}^{\infty} (n)(n-1) a_n t^{n-2}, \text{ so}
\]
\[
t^2 \frac{d^2 y}{dt^2} = \sum_{n=2}^{\infty} (n)(n-1) a_n t^n = \sum_{n=0}^{\infty} (n)(n-1) a_n t^n
\]

In this last series we have changed the starting index from \( n = 2 \) to \( n = 0 \) because

(i) including the two extra terms doesn’t change the sum since \( n(n-1) = 0 \) both for \( n = 0 \) and for \( n = 1 \), and

(ii) we wish to make the series for \( t^2 \frac{d^2 y}{dt^2} \) start with the same index as the series that we will use for \( y \) and for \( \frac{d^2 y}{dt^2} \): as in question 5, we will use
\[
\frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.
\]

Then
\[
(1 + t^2) \frac{d^2 y}{dt^2} + y = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n)(n-1)a_n + a_n] t^n,
\]

which we want to equal just \( 3t \). By equating the powers of \( t \),

\[
\begin{align*}
2a_2 + a_0 &= 0 \\
6a_3 + a_1 &= 3
\end{align*}
\]

\( (n+2)(n+1) a_{n+2} + (n^2 - n + 1)a_n = 0 \) for \( n \geq 2 \)

From the initial condition \( y(0) = 1 \), it follows that \( a_0 = 1 \) and then \( a_2 = -\frac{1}{2} a_0 = -\frac{1}{2} \). By the recurrence relation for \( n = 2 \), \( a_4 = -\frac{2^2 - 2^2 + 1}{(4)(3)} a_2 = +\frac{1}{2} \).

From the initial condition \( \frac{dy}{dt}(0) = 0 \) it follows that \( a_1 = 0 \) and then \( a_3 = \frac{3-a_1}{6} = \frac{1}{2} \). By the recurrence relation for \( n = 3 \), \( a_5 = -\frac{3^2 - 3^2 + 1}{(5)(4)} a_3 = -\frac{7}{30} \).

Therefore the particular solution satisfying these initial conditions is
\[
y = \left[ 1 - \frac{1}{2} t^2 + \frac{1}{8} t^4 + \ldots \right] \\
+ \left[ \frac{1}{2} t^3 - \frac{7}{30} t^5 + \ldots \right]
\]

In this example, since \( n^2 - n + 1 \) doesn’t factor, it is a bit messy to write out the general term and we shall leave the solution in the form shown.
13. The DE is of the form \( \frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0 \) where \( P(t) = -t \) and \( Q(t) = t^2 \). Since polynomials are analytic at every point, both of \( P \) and \( Q \) are analytic at \( t = 0 \) and so \( t = 0 \) is an ordinary point for the DE, and so cannot be a regular singular point.

15. We try
\[
y = \sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 = 1.
\]
Then
\[
\frac{dy}{dt} = \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1},
\]
\[
\frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2}.
\]
so
\[
t^2 \frac{d^2 y}{dt^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r}
\]
\[
-t \frac{dy}{dt} = \sum_{n=0}^{\infty} -(n+r)a_n t^{n+r},
\]
and
\[
t^2 \frac{d^2 y}{dt^2} - t \frac{dy}{dt} + y = \sum_{n=0}^{\infty} [(n+r)(n+r-1)a_n - (n+r)a_n + a_n] t^{n+r}
\]
which we want to equal 0. This will be true if the coefficients of the powers \( t^{n+r}, n = 0, 1, 2, \ldots \) all vanish. For \( n = 0 \), we must have
\[
(r)(r-1) - (r) + 1 = 0.
\]
This is the 
**indicial equation**. The roots \( r \) of \( r^2 - 2r + 1 = 0 \) are \( r_1 = r_2 = 1 \), so the method of Frobenius gives us a solution of the form
\[
y_1 = \sum_{n=0}^{\infty} a_n t^{n+1}
\]
where \( a_0 = 1 \) and the coefficients \( a_n \) for \( n \geq 1 \) must satisfy
\[
[(n+r_1)(n+r_1-1) - (n+r_1) + 1] a_n = 0, \; i.e.
\]
\[
[n^2] a_n = 0.
\]
Therefore \( a_n = 0 \) for \( n \geq 1 \), and \( y_1 = t \).