MATH 306C
Test 2
Mar. 31, 2017

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

• Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

• Please put your simplified final answers in the spaces provided.

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
<th>SCORE</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
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<td>10</td>
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<td>TOTAL</td>
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</tbody>
</table>
1. (10 points) a. 4pt Find the general solution of \( y'' - 4y' + 4y = 0 \).

\[
\begin{align*}
1 & \quad y'' - 4y' + 4y = 0 \\
2 & \quad r^2 - 4r + 4 = 0 \\
3 & \quad (r - 2)(r - 2) = 0 \\
4 & \quad r = 2 \quad \text{single, repeated root} \\
5 & \quad y = C_1e^{2t} + C_2te^{2t}
\end{align*}
\]

b. 4pt Solve the initial value problem \( y'' - 4y' + 4y = 0 \), \( y(0) = 1 \), \( y'(0) = 1 \).

\[
\begin{align*}
y &= C_1e^{2t} + C_2te^{2t} \\
1 &= C_1e^0 + C_2(0)e^0 \\
1 &= C_1 \\
y' &= 2C_1e^{2t} + C_2e^{2t} + 2C_2te^{2t} \\
1 &= 2(1)e^0 + C_2e^0 + 2C_2(0)e^0 \\
1 &= 2 + C_2 \\
-1 &= C_2 \\
1 &= C_1e^{2t} - C_2te^{2t}
\end{align*}
\]

\[
\begin{align*}
y &= e^{2t} - te^{2t}
\end{align*}
\]

c. 2pt Find \( \lim_{t \to \infty} y(t) \), where \( y(t) \) is the solution in b).

\[
\begin{align*}
y_1 &= e^{2t} \\
y_2 &= -te^{2t} \quad \text{outgrows } y_1 \\
\lim_{t \to \infty} -te^{2t} &= -\infty
\end{align*}
\]
2. (10 points) a. 4pt Find the form of a particular solution $y_p$ with the fewest constants $A, B, C, \ldots$ for $y'' - 4y' + 3y = (t^2 + 1)e^t$.

$$r^2 - 4r + 3 = 0$$

$$(r - 1)(r - 3) = 0$$

$r = 1 \quad r = 3$

$y_1 = C_1 e^t$$

$y_2 = C_2 e^{3t}$

$y_p = (At^2 + Bt + C)e^t$

however, $Ce^t$ is a solution of the homogeneous equation, so must multiply by $t$.

$y_p = (At^3 + Bt^2 + Ct)e^t$

b. 6pt The differential equation $y'' + y = \sin(t)$ has a particular solution of the form $y_p = At \cos t + Bt \sin t$. Find $y_p$.

$$y_p = At \cos t + Bt \sin t$$

$$y_p' = Acos t - Asint + Bsint + Bcost$$

$$y_p'' = -Asint - Asint - Acos t + Bcost + Bcost - Btsint$$

$$-2Asint - Atcos t + 2Bcost - Btsint + Atcos t + Btsint = \sin t$$

$$-2Asint + 2Bcost = \sin t + \cos t$$

$$-2Asint = \sin t$$

$$2Bcost = \cos t$$

$-2A = 1$

$A = -\frac{1}{2}$

$2B = 0$

$B = 0$

$$y_p = -\frac{1}{2} t \cos t$$
3. (10 points) An object with mass \( m = 3 \) kg stretches a spring \( 10/9 \) meters to its equilibrium position. Assume that the acceleration due to gravity is \( g = 10 \) meter/sec\(^2\). The spring constant \( k \) is then such that \( k \frac{10}{g} = mg \). No damping device is attached.

a. 2pt Write down a differential equation for \( y(t) \), the displacement of the object from its equilibrium position. \( y > 0 \) means the object is below equilibrium position and \( \frac{dy}{dt} > 0 \) means the object is moving downwards.

\[ 3y'' + 27y = 0 \]

b. 2pt At time \( t = 0 \) the object is released 1 meters below its equilibrium position with a downward velocity of 3 meters/sec. Write down the initial conditions.

\[ y(0) = 1 \quad y'(0) = 3 \]

c. 3pt Find \( y(t) \), the solution of the above initial value problem.

\[ 3(r^2 + 9) \quad y(t) = C_1 \cos 3t + C_2 \sin 3t \]

\[ r^2 = -9 \quad y(0) = C_1 = 1 \]

\[ r = \pm 3i \quad y'(0) = 3C_1 \sin 0 + 3C_2 \cos 0 = 3 \]

\[ C_2 = 3 \]

\[ y(t) = \cos 3t + \sin 3t \]

d. 3pt Find an expression of the form \( y(t) = C \cos(\omega t - \alpha) \) for the solution.

\[ y(t) = \sqrt{2} \cos \left( 3t - \frac{\pi}{4} \right) \]
4. (10 points) A mass-spring-damper system is described by

\[ mx''(t) + cx'(t) + kx(t) = 0 \]

where mass \( m = 1 \), spring constant \( k = \frac{5}{4} \) and damping coefficient \( c > 0 \).

a. For what values of \( c \) are the roots \( r \) of the characteristic equation imaginary?

\[ r = \frac{-c \pm \sqrt{c^2 - 4\left(\frac{5}{4}\right)}}{2} \]

\( \sqrt{c^2 - 5} < 0 \) \quad when \quad \( 0 < c < \sqrt{5} \)

b. For the value \( c = 1 \), find a general solution \( x(t) \) in terms of real functions.

\[ r^2 + r + \frac{5}{4} = 0 \]

\[ r = \frac{-1 \pm \sqrt{1 - 5}}{2} \]

\[ y = c_1 e^{-\frac{1}{2}t} \cos t + c_2 e^{-\frac{1}{2}t} \sin t \]

c. For solutions \( x(t) \) for \( c = 1 \), find the pseudo-period \( T \). Note: The interval between times at which \( x(t) \) passes through 0, is \( T/2 \).

\[ y = c e^{-\frac{1}{2}t} \cos(t - \alpha) \]

\[ \frac{\pi}{2} = t_1 - \alpha \]

\[ \frac{3\pi}{2} = t_2 - \alpha \]

\[ \pi = t_1 - t_2 \]

\[ \pi = t_2 - t_1 \]

\[ T = \frac{T}{2} \]

\[ T = 2\pi \]
5. **(10 points)** Match each differential equation with the best description of long time solution behavior. (Please place your answer I, II, III .. in the space provided below the equation.)

a. \( y'' + 4y = 0 \)

\[
\begin{align*}
   r^2 &= 4 \\
   r &= \pm 2 \\
   c_1 \cos 2t, c_2 \sin 2t
\end{align*}
\]

\[\boxed{\text{VI}}\]

- I. Every solution approaches 0 as \( t \to \infty \)

b. \( y'' - 8y' + 7y = 0 \)

\[
\begin{align*}
   r^2 - 8r + 7 &= 0 \\
   (r - 7)(r - 1) &= 0 \\
   r &= 7, 1 \\
   c_1 e^{7t}, c_2 e^t
\end{align*}
\]

\[\boxed{\text{III}}\]

- II. Has a nonzero solution that approaches 0 as \( t \to \infty \) and has a nonzero solution that approaches \( \infty \) as \( t \to \infty \).

- III. Every nonzero solution approaches either \( \infty \) or \(-\infty\) as \( t \to \infty \)

- IV. Every nonzero solution has oscillations which become progressively larger as \( t \to \infty \)

- V. Every nonzero solution has oscillations which become progressively smaller as \( t \to \infty \)

- VI. Every nonzero solution oscillates with constant amplitude as \( t \to \infty \)

d. \( y'' + 2y' + y = 0 \)

\[
\begin{align*}
   r^2 + 2r + 1 &= 0 \\
   (r + 1)^2 &= 0 \\
   r &= -1, -1 \\
   c_1 e^{-t}, c_2 e^{-t}
\end{align*}
\]

\[\boxed{\text{I}}\]

e. \( y'' - 6y' + 13y = 0 \)

\[
\begin{align*}
   r^2 - 6r + 13 &= 0 \\
   \left( r - 3 \pm \sqrt{36 - 52} \right) &= 0 \\
   r &= 3 \pm 2i \\
   c_1 e^{3t} \cos 2t + c_2 e^{3t} \sin 2t
\end{align*}
\]

\[\boxed{\text{IV}}\]