MATH 306C
Test 1
Friday, Feb. 24 2017

NAME (please print legibly): INSTRUCTOR
Your University ID Number:

• Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

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<tr>
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<td><strong>TOTAL</strong></td>
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</table>
1. (10 points)

a. (8pts) Find the general solution to the following ODE

\[ x \frac{dy}{dx} + 2y = 4, \ x > 0 \]

\[ \frac{dy}{dx} + \frac{2}{x} y = \frac{4}{x} \text{, first order linear} \]

\[ P = \frac{2}{x} \]

\[ \int P \, dx = \int \frac{2}{x} \, dx = 2 \ln x \]

\[ \psi = e^{\int P \, dx} = e^{2 \ln x} = x^2 \]

\[ \text{Multipliers DE: } x^2 \frac{dy}{dx} + 2xy = 4x \]

\[ \frac{d}{dx} (x^2 y) = 4x \]

\[ x^2 y = 2x^2 + C \]

\[ y = 2 + \frac{C}{x^2} \]

b. (2pts) Find the solution of the above equation which satisfies \( y(1) = 1 \).

\[ y(1) = 2 + \frac{C}{1} = 1 \Rightarrow C = 1 - 2 = -1 \]

\[ y(x) = 2 - \frac{1}{x^2} \]
2. (10 points)

a. (3pts) Verify that the following ODE is exact:

\[
\begin{align*}
2x + e^y + (xe^y + 3y^2) \frac{dy}{dx} &= 0 \\
M &= 2x + e^y \\
N &= xe^y + 3y^2
\end{align*}
\]

(Show all work.)

\[
\begin{align*}
\frac{\partial M}{\partial y} &= \frac{\partial^2 (2x + e^y)}{\partial y} = 0 + e^y \\
\frac{\partial N}{\partial x} &= \frac{\partial^2 (xe^y + 3y^2)}{\partial x} = xe^y + 3y^2
\end{align*}
\]

\[\text{Since } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ exact}\]

b. (5pts) Find the general solution to the ODE in part a). (You may leave your answer in implicit form.)

\[
\begin{align*}
\text{Find } F: \quad &\frac{\partial F}{\partial x} = 2x + e^y \\
&\frac{\partial F}{\partial y} = xe^y + 3y^2
\end{align*}
\]

\[F = x^2 + xe^y + h(y)\]

Also

\[
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} (x^2 + xe^y + h(y)) = xe^y + 3y^2
\]

\[
x^2 + h'(y) = xe^y + 3y^2
\]

\[2(y) = y^3 + C\]

\[F = x^2 + xe^y + y^3 + C\]

Doesn't matter if \(C\) in part c.

c. (2pts) Find the solution to the ODE in part a) which satisfies \(y(0) = 1\). (You may leave your answer in implicit form.)

\[x^2 + xe^y + y^3 = C\]

Also \(y(0) = 1 \Rightarrow 0 + 0e^0 + 1^3 = C \Rightarrow C = 1\).

\[x^2 + xe^y + y^3 = 1\]
3. (10 points) A tank initially contains 100 liters of salt solution. Salt solution with concentration 1 gram per liter flows into the tank, at 20 liters per minute. In the tank the solution is well mixed, and solution is drained from the tank at 20 liters per minute. Let \( x = x(t) \) be the amount (in grams) of salt in the tank at time \( t \) minutes.

a. (2pts) What is the volume \( V \) of solution in the tank, and what is the concentration in grams per liter, of the solution that leaves the tank? (leave your answer for the concentration, in terms of \( x(t) \) and the volume \( V \))

\[
\text{Not } V = 100 \text{ liters}, \quad C_0 = \frac{x(t)}{V} \text{ grams/liter leaves.}
\]

b. (2pts) Find a formula that approximates \( \Delta x \), the change in the amount of salt over the time interval \([t, t + \Delta t] \), in terms of \( x \) and \( \Delta t \), the incremental change in time.

\[
\Delta x = (C_t x_t - C_0 x_0) \Delta t = \left( \frac{x(t)}{V} \right) (20) \Delta t
\]

c. (2pts) Write a differential equation for the amount of salt in the tank.

\[
\frac{dx}{dt} = 20 - \frac{20}{100} x \quad 1/\text{pt}
\]

\[
\mu = e^t \quad 1/\text{pt}
\]

\[
\frac{d}{dt} \left( e^{t/5} x(t) \right) = 20 e^{t/5} \quad 1/\text{pt}
\]

\[
e^{t/5} x(t) - 70 = 20 \int_0^t e^{s/5} ds = 20 \frac{e^{t/5}}{1/5} \bigg|_0^t = 100 \left( e^{t/5} - 1 \right)
\]

\[
x(t) = 70 e^{-t/5} + 100 - 100 e^{-t/5}
\]

\[
= 100 - 30 e^{-t/5}
\]

d. (3pts) Solve for \( x(t) \), assuming that \( x(0) = 70 \) grams.

\[
\lim_{t \to \infty} x(t) = 100 - 0 = 100.
\]
4. (10 points) Consider the differential equation

\[ x'(t) = -4x + x^3 \]

\[ f(x) \]

a. (3pts) Determine all the equilibrium solutions of this ODE.

\[ f(x) = 0 \Rightarrow -4x + x^3 = 0 \]
\[ x(-4 + x^2) = 0 \]
\[ x = 0 \quad \text{or} \quad x^2 = 4 \]
\[ x = \pm 2 \]

\[ x = 0, \quad x = +2, \quad x = -2 \] are the equilibrium solutions.

b. (5pts) Sketch a phase diagram for this ODE.

\[ 1^{st} \text{ per.} \]
\[ 1^{st} \text{ for line} \]
\[ \text{with} \]
\[ \text{equilibria,} \]
\[ \text{1st per arrow} \]

\[ \text{if } x > 2, f = \frac{x(x^2 - 4)}{x} > 0 \quad \text{so } \uparrow \]
\[ \text{if } 0 < x < 2, f = \frac{x(x^2 - 4)}{x} > 0 \quad \text{so } \uparrow \]
\[ \text{if } -2 < x < 0, f = \frac{x(x^2 - 4)}{x} > 0 \quad \text{so } \uparrow \]
\[ \text{if } x < -2, f = \frac{x(x^2 - 4)}{x} < 0 \quad \text{so } \downarrow \]

\[ 2 \cdot U \]
\[ 0 \cdot S \]
\[ -2 \cdot U \]

\[ \text{if } x < -2, f = \frac{x(x^2 - 4)}{x} < 0 \quad \text{so } \downarrow \]
\[ \text{if } x < -2, f = \frac{x(x^2 - 4)}{x} < 0 \quad \text{so } \downarrow \]

\[ \text{1st per arrow} \]
\[ \text{1st for line} \]
\[ \text{if } x > 2, f = \frac{x(x^2 - 4)}{x} > 0 \quad \text{so } \uparrow \]
\[ \text{if } 0 < x < 2, f = \frac{x(x^2 - 4)}{x} > 0 \quad \text{so } \uparrow \]
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\[ \text{if } x < -2, f = \frac{x(x^2 - 4)}{x} < 0 \quad \text{so } \downarrow \]

\[ \text{c. (2pts) For each equilibrium solution found in Part a, determine whether it is stable or unstable and mark each equilibrium on the phase diagram as either "S" or "U".} \]

\[ \text{\underline{x = 0 is stable}, \quad \underline{x = 2 is unstable}, \quad \underline{x = -2 is unstable}} \]

\[ \underline{1\text{pt}} \]
\[ \underline{1\text{pt}} \]
5. (10 points) Consider the DE \( \frac{dy}{dx} = 4xy^{\frac{1}{2}} \)

a. (3pts) Solve by separation of variables. Write \( C \) for the constant of integration.

\[
\int y^{-\frac{1}{2}} dy = 4x \int dx \\
\frac{y^{-\frac{1}{2}} + 1}{\frac{1}{2} + 1} = 2x^2 + C \\
2y^{\frac{1}{2}} = 2x^2 + C \\
y = \left(\frac{x^2 + \frac{C}{2}}{2}\right)^2.
\]

b. (2pts) Find \( C \) such that the solution from part a) passes through \((x, y) = (0, 0)\) and write out the solution \( y(x) \).

\[
y(0) = \left(0 + \frac{0}{2}\right)^2 = 0 \\
C = 0 \\
y(x) = \left(x^2 + \frac{0}{2}\right)^2 = x^4.
\]

c. (2pts) Find another solution of the same initial value problem \( \frac{dy}{dx} = 4xy^{\frac{1}{2}} \), \( y(0) = 0 \).

By trying the formula \( y(x) \equiv 0 \), find: \( \frac{dy}{dx} = 4x(0^{\frac{1}{2}}) \), also \( y(0) = 0 \). \( y(x) \equiv 0 \) is another solution.

d. (3pts) Write the DE in the form, \( \frac{dy}{dx} = F(x, y) \) and calculate \( \frac{\partial F}{\partial y} \). For what points \((a, b)\) is it guaranteed that there exists a unique solution that passes through the point \((x, y) = (a, b)\)?

\[
F(x, y) = 4xy^{\frac{1}{2}} \\
\frac{\partial F}{\partial y} = \frac{3}{2} 4x y^{\frac{1}{2}} = 2x^{\frac{3}{2}} y^{\frac{1}{2}} = \frac{4x}{\sqrt{y}}.
\]

If \( b > 0 \), then for any \( a \), both \( F, \frac{\partial F}{\partial y} \) are continuous in a rectangle containing \((a, b)\). Therefore, there exists a unique solution.