MATH 306C
Test 2
Oct. 23, 2015

NAME (please print legibly): **INSTRUCTOR SOLUTIONS**
Your University ID Number: ________________________________

- Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.
- Please put your simplified final answers in the spaces provided.

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
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<tr>
<td>2</td>
<td>10</td>
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<td>3</td>
<td>10</td>
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<td>4</td>
<td>10</td>
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<td>5</td>
<td>10</td>
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<td><strong>TOTAL</strong></td>
<td><strong>50</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points)  
   a. 4pt Find the general solution of \( y'' - 2y' + y = 0 \).

   Try \( y = e^{rt} \).  
   \((r^2 - 2r + 1)e^{rt} = 0\)

   \((r-1)(r-1) = 0\); \( r = 1 \) is a double root

   1.I. Solve  \( y_1 = e^{rt}, \ y_2 = te^{rt} \)

   General solution  \( y = c_1 e^{rt} + c_2 te^{rt} \)

   b. 4pt Solve the initial value problem \( y'' - 2y' + y = 0 \), \( y(0) = -1, y'(0) = 1 \).

   \[
   \begin{align*}
   \text{at } t = 0, \quad & y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = -1 \quad \overset{1\text{pt}}{=} \quad (1) \\
   y'(t) &= c_1 e^t + c_2 (e^t + te^t) \\
   y'(0) &= c_1 e^0 + c_2 (e^0 + 0e^0) = 1 \quad \overset{1\text{pt}}{=} \quad (2)
   \end{align*}
   \]

   From (1), \( c_1 = -1 \).

   Then from (2), \( c_2 = 1 - c_1 = 1 - (-1) = 2 \)

   Solution  \( y(t) = -e^t + 2te^t \)

   c. 2pt Find \( \lim_{t \to \infty} y(t) \), where \( y(t) \) is the solution in b).

   \( y(t) = -e^t + 2te^t \)

   For \( t \) large, \(-1 + 2t\) is large positive.

   Any explanation  \( \overset{1\text{pt}}{=} \)

   Also, \( \lim_{t \to \infty} e^t = +\infty \) and \( \lim_{t \to \infty} (-1 + 2t) = +\infty \)

   \( \therefore \lim_{t \to \infty} y(t) = +\infty \)

   1pt
2. (10 points) a. 4pt Find the form of a particular solution $y_p$ with the fewest constants $A$, $B$, $C$, .. for

$$y'' - 4y = (t^2 + t + 1)e^{2t}$$

First guess is $y_p = (At^2 + Bt + C)e^{2t}$.

However, solutions of homogeneous eqn are $y_1 = e^{2t}$, $y_2 = e^{-2t}$. The guess must not include solution of hom. eqn.

i. multiply by $t$: $y_p = (At^3 + Bt^2 + Ct)e^{2t}$

If $(At^2 + Bt + C)e^{2t}$, 1 pt

2 pt / 4

b. 6pt The differential equation $y'' + y = 2\cos(t)$ has a particular solution of the form $y_p = At\cos t + Bt\sin t$. Find $y_p$.

$$y = At\cos t + Bt\sin t$$

$$y' = A(t(-\sin t) + \cos t) + B(t(\cos t) + \sin t)$$

$$y'' = A(t(-\cos t) - \sin t - \sin t) + B(t(-\sin t + \cos t) + \cos t + \cos t)$$

$$y'' + y = A(2\sin t) + B(2\cos t) = 2\cos t + 0\cdot\sin t$$

Require $A = 0$, 1 pt

$B = 1$, 1 pt

Solution is $y_p = t\sin t$. 
3. (10 points) An object with mass \( m = 9 \) kg stretches a spring 10/9 meters to its equilibrium position. Assume that the acceleration due to gravity is \( g = 10 \) meter/sec\(^2\). The spring constant \( k \) is then such that \( k \frac{10}{9} = mg \). No damping device is attached.

a. 2pt Write down a differential equation for \( y(t) \), the displacement of the object from its equilibrium position. \( y > 0 \) means the object is below equilibrium position and \( \frac{dy}{dt} > 0 \) means the object is moving downwards.

\[
\frac{d^2y}{dt^2} + 81y = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} + 9y = 0
\]

b. 2pt At time \( t = 0 \) the object is released 1 meters above its equilibrium position with a downward velocity of 3 meters/sec. Write down the initial conditions.

\[
y(0) = -1 \quad \text{and} \quad y'(0) = 3
\]

c. 3pt Find \( y(t) \), the solution of the above initial value problem.

\[
\frac{d^2y}{dt^2} + 9y = 0 \quad \Rightarrow \quad z^2 + 9 = 0 \quad \Rightarrow \quad z = \pm \sqrt{-9} = \pm 3i
\]

\[
y = c_1 \cos 3t + c_2 \sin 3t
\]

\[
y(0) = 6 \quad \Rightarrow \quad c_1 \cdot 1 + c_2 \cdot 0 = -1 \quad \Rightarrow \quad c_1 = -1
\]

\[
y'(0) = -3c_1 \sin 3t + 3c_2 \cos 3t
\]

\[
y'(0) = -3(-1) \cdot 0 + 3c_2 \cdot 1 = 3 \quad \Rightarrow \quad c_2 = 1
\]

d. 3pt Find an expression of the form \( y(t) = C \cos(\omega t - \alpha) \) for the solution.

\[
y(t) = \sqrt{2} \cos (3t - \frac{3\pi}{4})
\]

\[
\text{ANSWER: } \quad \boxed{y(t) = \sqrt{2} \cos (3t - \frac{3\pi}{4})}
\]

1pt
4. (10 points) A mass-spring-damper system is described by

\[ m\ddot{x}(t) + cx'(t) + kx(t) = 0 \]

where the mass \( m = 1 \), damping coefficient \( c = 2 \) and spring constant \( k > 0 \).

a. For what values of \( k \) are the roots \( r \) of the characteristic equation real?

char. eqn \( 1 \cdot r^2 + 2 \cdot r + k = 0 \)

\[ r = \frac{-2 \pm \sqrt{4 - 4k}}{2} = -1 \pm \sqrt{1 - k} \]

Roots \( r \) are real if \( 1 - k \geq 0 \); \( k \leq 1 \)

b. For the value \( k = 5 \), find a general solution \( x(t) \) in terms of real functions.

\[ r = -1 \pm \sqrt{1 - 5} = -1 \pm \sqrt{-4} = -1 \pm 2i \]

\[ x(t) = A e^{-t} \cos 2t + B e^{-t} \sin 2t \]

c. For solutions \( x(t) \) for \( k = 5 \), find the pseudo-period \( T \). Note: The interval between times at which \( x(t) \) passes through 0, is \( T/2 \).

If convert to \( x(t) = C e^{-t} \cos(2t - \alpha) \),

\[ \cos(\theta) = 0 \text{ for } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \]

\[ 2t - \alpha = \frac{\pi}{2} \]

\[ 2t_2 - \alpha = \frac{3\pi}{2} \]

\[ 2(t_2 - t_1) = \pi \text{; } t_2 - t_1 = \frac{\pi}{2} \]

\[ T = t_2 - t_1 = \frac{\pi}{2} \Rightarrow T = \pi \]
5. (10 points) Match each differential equation with the best description of long time solution behavior. (Please place your answer I, II, III... in the space provided below the equation.)

a. \( y'' + y' + \frac{1}{4} y = 0 \)
\[
\begin{align*}
\lambda^2 + \lambda + \frac{1}{4} &= 0 \\
(\lambda + \frac{1}{2})^2 &= 0 \\
\lambda &= -\frac{1}{2}, \quad e^{-\frac{1}{2}t}, \quad te^{-\frac{1}{2}t} \\
&\text{I}
\end{align*}
\]

b. \( y'' - 2y' + 5y = 0 \)
\[
\begin{align*}
\lambda^2 - 2\lambda + 5 &= 0 \\
\lambda &= \left(2 \pm \sqrt{4 - 20}\right)/2 \\
\lambda &= 1 \pm 2i \\
e^t \cos 2t, e^t \sin 2t &\text{ IV}
\end{align*}
\]

c. \( y'' + 9y = 0 \)
\[
\begin{align*}
\lambda^2 + 9 &= 0 \\
\lambda &= \pm 3i \\
\cos 3t, \sin 3t &\text{ V I}
\end{align*}
\]

d. \( y'' - 5y' + 4y = 0 \)
\[
\begin{align*}
\lambda^2 - 5\lambda + 4 &= 0 \\
(\lambda - 1)(\lambda - 4) &= 0 \\
e^t, e^{4t} &\text{ III}
\end{align*}
\]

e. \( y'' + y' - 2y = 0 \)
\[
\begin{align*}
\lambda^2 + \lambda - 2 &= 0 \\
(\lambda - 2)(\lambda + 1) &= 0 \\
e^{-2t}, \quad e^t &\text{ II}
\end{align*}
\]

I. Every solution approaches 0 as \( t \to \infty \)

II. Has a nonzero solution that approaches 0 as \( t \to \infty \) and has a nonzero solution that approaches \( \infty \) as \( t \to \infty \).

III. Every nonzero solution approaches either \( \infty \) or \( -\infty \) as \( t \to \infty \).

IV. Every nonzero solution has oscillations which become progressively larger as \( t \to \infty \).

V. Every nonzero solution has oscillations which become progressively smaller as \( t \to \infty \).

VI. Every nonzero solution oscillates with constant amplitude as \( t \to \infty \).

For each of a) - e):
2 pts if correct,
0 pts if incorrect and no solutions given,
1 pt if incorrect but one solution correct.

ANSWER: __________________________