MATH 306C

Test 1
Sept. 25, 2015

NAME (please print legibly): ______________
Your University ID Number: ____________________

- Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.
- Please put your simplified final answers in the spaces provided.

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1. (10 points)

a. (8pts) Find the general solution to the following ODE

\[ \frac{dy}{dx} - \frac{2}{x} y = \frac{3}{x}, \quad x \geq 0 \]

**2pts** Standard form:

\[ \frac{dy}{dx} - \frac{2}{x} y = \frac{3}{x} \]

**4pts** Integrating factor:

\[ e^{\int \frac{-2}{x} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2} \]

**2pts** Multiply by \( x^{-2} \) gives:

\[ x^{-2} \frac{dy}{dx} - 2x^{-3} y = 3x^{-3} \]

**1pt** The left side is:

\[ \frac{d}{dx} \left( x^{-2} y \right) = 3x^{-3} \]

**2pts** Integrating:

\[ x^{-2} y = -\frac{3}{2} x^{-2} + C \]

**2pts** Multiply by \( x^2 \) gives:

\[ y = -\frac{3}{2} + C x^2 \]

b. (2pts) Find the solution of the above equation which satisfies \( y(1) = 0 \).

**1pt** For \( x = 1 \),

\[ y(1) = -\frac{3}{2} + C \cdot 1^2 = 0 \]

so \( C = \frac{3}{2} \)

and

\[ y(x) = -\frac{3}{2} + \frac{3}{2} x^2 \]

**1pt** ANSWER: 

\[ y(x) = -\frac{3}{2} + \frac{3}{2} x^2 \]
2. (10 points)

a. (3pts) Verify that the following ODE is exact:

\[ 3x^2 + \sin y + (2y + x \cos y) \frac{dy}{dx} = 0 \]

(Show your work.)

\[ \begin{align*}
\text{LHF} & \quad D_y M = 0 + \cos y \\
\text{RHF} & \quad D_x N = 0 + 1 \cdot \cos y \\
& \quad \text{Since } D_y M = D_x N, \ ODE \text{ is exact}
\end{align*} \]

b. (5pts) Find the general solution to the ODE in Part a. (You may leave your answer in implicit form.)

Want \( F(x, y) \) such that \( D_x F = M \) and \( D_y F = N \).

\[ \begin{align*}
D_x F &= 3x^2 + \sin y \\
F &= x^3 + x \sin y + h(y) \\
\text{(integrate regarding } y \text{ as constant.)}
\end{align*} \]

\[ \begin{align*}
D_y F &= 0 + x (\cos y) + h'(y) \\
& \quad \text{want } D_y F = 2y + x \cos y
\end{align*} \]

\[ h'(y) = 2y \]

\[ h(y) = y^2 + C \]

\[ F = x^3 + x \sin y + y^2 + C \] or \[ x^3 + x \sin y + y^2 = \text{constant} = C_1 \]

c. (2pts) Find the solution to the ODE in Part a. which satisfies \( y(0) = 1 \). (You may leave your answer in implicit form.)

\[ \text{If } y(0) = 1, \text{ then } 0 + 1 \cdot \sin 1 + 1 = C_1 \]

\[ C_1 = 1 \]

\[ 20 \quad x^3 + x \sin y + y^2 = 1 \] is solved in implicit form.
3. (10 points) A tank initially contains 10 liters of pure water. Salt solution with concentration 5 grams per liter flows into the tank, at 3 liters per minute. In the tank the solution is well mixed, and is drained from the tank at 3 liters per minute.

Let \( x = x(t) \) be the amount (in grams) of salt in the tank at time \( t \) minutes.

a. (2pt) What is the volume of solution in the tank, and what is the concentration in grams per liter, of the solution that leaves the tank?

\[
\text{Volume} = 10 \quad \text{and concentration} \quad \frac{x(t)}{10} \quad \text{grams/liter}
\]

b. (2pts) Find a formula that approximates \( \Delta x \), the change in the amount of salt over the time interval \([t, t + \Delta t]\), in terms of \( x \) and \( \Delta t \), the incremental change in time.

\[
\Delta x = 5 \frac{\text{gm}}{\text{min}} \cdot 3 \frac{\text{liters}}{\text{min}} \cdot \Delta t - \frac{x(t) \text{gm}}{10 \text{liters}} \cdot 3 \frac{\text{liters}}{\text{min}} \cdot \Delta t
\]

c. (2pts) Write a differential equation for the amount of salt in the tank.

\[
\frac{dx}{dt} = 15 - \frac{3}{10} x
\]

d. (3pts) Solve for \( x(t) \).

\[
\frac{dx}{dt} + \frac{3}{10} x = 15 \quad \rightarrow \quad e^{\frac{3}{10} t} \frac{dx}{dt} = 15 e^{\frac{3}{10} t} \quad \rightarrow \quad \frac{d}{dt} \left( e^{\frac{3}{10} t} x \right) = 15 e^{\frac{3}{10} t} \quad \rightarrow \quad e^{\frac{3}{10} t} x = 15 \left( e^{\frac{3}{10} \cdot t} \right) \bigg|_0^t \quad \rightarrow \quad x(t) = 50 \left( 1 - e^{-\frac{3}{10} t} \right)
\]

e. (1pts) Calculate the limit \( \lim_{t \to \infty} x(t) \).

\[
\text{As } t \to \infty, \quad e^{-\frac{3}{10} t} \to 0 \quad \Rightarrow \quad x(t) \to 50
\]

\[1 \text{pt}\]
4. (10 points) Consider the differential equation

\[ x'(t) = x^3 - x^2 \]

a. (3pts) Determine all the equilibrium solutions of this ODE.

\[ x^2(x-1) = 0 \Rightarrow x = 0 \quad \text{or} \quad x = 1 \]

\[ \text{let for factoring} \]

b. (5pts) Sketch a phase diagram for this ODE.

\[ \begin{align*}
\text{If } x > 1, & \quad x' = x^2(x-1) > 0 & \Rightarrow & \quad 1 \quad \text{(let)} \\
\text{If } 0 < x < 1, & \quad x-1 < 0 \quad \text{and} \quad x' = x^2(x-1) < 0 & \Rightarrow & \quad 0 \quad \text{(let)} \\
\text{If } x < 0, & \quad x^2 > 0 \quad \text{and} \quad x-1 < 0, & \Rightarrow & \quad 0 \quad \text{(let)} \\
\end{align*} \]

\[ \text{let for line with equilibria} \]

c. (2pts) For each equilibrium solution found in Part a, determine whether it is stable or unstable.

\[ x = 1 \quad \text{is unstable} \]
\[ x = 0 \quad \text{is unstable} \]
5. (10 points) Consider the DE $\frac{dy}{dx} = 3xy^{\frac{1}{3}}$

a. (3pts) Solve by separation of variables. Write $C$ for the constant of integration.

\[
\begin{align*}
\int y^{-\frac{1}{3}} \, dy &= 3 \int x \, dx \\
\frac{2}{\sqrt[3]{3}} y^{\frac{2}{3}} + C_1 &\Rightarrow y^{\frac{2}{3}} = \frac{2}{\sqrt[3]{3}} x^2 + C_1 \\
\text{where } C = \frac{2}{\sqrt[3]{3}} C_1
\end{align*}
\]

\[
\begin{align*}
y &= \left( \frac{2}{\sqrt[3]{3}} x^2 + C_1 \right)^{\frac{3}{2}} \\
p\text{et}
\end{align*}
\]

b. (2pts) Find $C$ such that the solution from Part a) passes through $(x, y) = (1, 0)$ and write out the solution $y(x)$.

\[
\begin{align*}
y(1) = 0 &\Rightarrow 0^{\frac{2}{3}} = \frac{2}{\sqrt[3]{3}} 1^2 + C_1 \\
&\Rightarrow C_1 = -\frac{2}{\sqrt[3]{3}} \\
y &= \left( \frac{2}{\sqrt[3]{3}} x^2 - 1 \right)^{\frac{3}{2}} \\
p\text{et}
\end{align*}
\]

c. (3pts) Write the DE in the form, $\frac{dy}{dx} = F(x, y)$ and calculate $\frac{\partial F}{\partial y}$. For what points $(a, b)$ is it guaranteed that there exists a unique solution that passes through the point $(x, y) = (a, b)$?

\[
\begin{align*}
\frac{dy}{dx} &= F(x, y) = 3xy^{\frac{1}{3}} \\
\frac{\partial F}{\partial y} &= 3x \cdot \frac{1}{3} y^{-\frac{2}{3}} = \frac{x}{y^{\frac{2}{3}}}
\end{align*}
\]

\[
\text{Any } (a, b) \text{ such that } x^{\frac{2}{3}} \text{ is defined and not zero} \Rightarrow a \neq 0
\]

p\text{et}

d. (2pts) Find another solution of the same initial value problem $\frac{dy}{dx} = 3xy^{\frac{1}{3}}$, $y(1) = 0$.

\[
\begin{align*}
\text{Guess: } y(x) &= 0 \\
p\text{et}
\end{align*}
\]

\[
\begin{align*}
\text{Check: } \frac{dy}{dx} &= 3x(0)^{\frac{1}{3}} = 0 \\
y(1) &= 0
\end{align*}
\]